Universidad Autónoma de San Luis Potosí

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Análisis de Perturbación de Eigenvalores para Sistemas con Retardo LTI. Casos Regulares y Singulares

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ALEJANDRO MARTÍNEZ GONZÁLEZ

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Directores de la Tesis:

César Fernando F. Méndez Barrios
Silviu-Iulian Niculescu

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Statement of Originality

I, Alejandro Martínez, declare that this thesis entitled ”Perturbation Analysis of Eigenvalues for LTI Delay Systems. Regular and Singular Cases” and the work presented in it are my own.

Abstract

Paris-Saclay Unibersity
CentraleSupélec School
Laboratory of Signals and Systems
Doctor of Philosophy:

Perturbation Analysis of Eigenvalues for LTI Delay Systems. Regular and Singular Cases

by MARTINEZ-GONZALEZ Alejandro

This dissertation is devoted to the analysis of the effects induced by the delays on the behavior of the dynamical systems described by linear delay-differential equations of retarded type including discrete delays in their mathematical representation. The main contributions of the thesis concern the characterization of the asymptotic behavior of multiple characteristic roots with respect to the delays in two configurations: one or two (delay) parameters. The proposed results and related algorithms give a better understanding of the underlying mechanisms (one or two delay parameters) and relax the existing conditions from the open literature (two delays, seen as parameters). To derive such criteria, the proposed approach combines the Weierstrass Preparation Theorem with the Newton Diagram Method. Finally, such ideas are also used to study the ill-posed/well-posed character of a closed-loop system when the derivative action is approximated by a delay-difference operator. In this last case study, the corresponding derived conditions are necessary and sufficient.

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Contents

List of Figures

List of Tables

Symbols

$\mathbb{N}$	The set of Natural Numbers
$\mathbb{Z}$	The set of Integer Numbers
$\mathbb{Z}_+$	The set of Positive Integer Numbers
$\mathbb{Q}$	The set of Rational Numbers
$\mathbb{R}$	The set of Real Numbers
$\mathbb{R}_+$	The set of Positive Real Numbers
$\mathbb{C}$	The set of Complex Numbers
$\Re$	The real part of a Complex Numbers
$\Im$	The real part of a Complex Numbers
$ord(\cdot)$	The order of a series
$card(\cdot)$	The Cardinality of a set
Δ	The discriminant of a polynomial
$\mathcal{C}$	A polyhedral cone
CRS	The Complete Regular Splitting property
RS	The Regular Splitting property
NRS	The Non-Regular Splitting property
$det(A)$	The Determinant of the matrix A
$\sigma(p)$	The spectrum of the polynomial p

Résumé étendu en français

Cette thèse a été réalisée à travers un travail de recherche scientifique qui explore et analyse les effets du retard sur la stabilité des systèmes linéaires. Une attention particulière est accordée au comportement des racines critiques de l'équation caractéristique du système. Pour faire face à la complexité du problème, du fait de la présence d'un retard comme paramètre, la continuité des racines est utilisée. Pour ce faire, l'espace des paramètres déterminé par le retard est pris, qui est divisé en régions stables et instables. Basé sur la continuité des racines; Les changements de stabilité peuvent être déterminés en analysant la tendance des racines aux limites de stabilité en utilisant le théorème de fonction implicite pour calculer la tendance à travers la frontière de stabilité.

Les objectifs essentiels abordés dans la thèse sont brièvement présentés ici. Description du comportement asymptotique de plusieurs racines à travers un algorithme, déploiement de la caractérisation de plusieurs racines à travers un seul critère et sous un schéma de contrôle particulier, décrire le comportement de racines singulières. Ainsi, la nouveauté de ce travail est de combler ces lacunes et d'élargir la méthodologie pour utiliser le même outil dans différentes situations, même dans des problèmes mal posés. On peut observer le mouvement des racines décrit par des trajectoires continues, qui s'effondrent sur l'axe pour former une racine multiple. C'est dans l'axe imaginaire que se concentre l'étude du comportement asymptotique où la multiplicité est supérieure à 1. Par conséquent, dans ce cas, la différentiabilité des racines est perdue, ce qui empêche l'utilisation du théorème de fonction implicite. Pour cette raison, il est nécessaire d'utiliser un autre outil pour l'étude de la duplication. Pour résoudre ce problème, une procédure algébrique systématique a été proposée pour déterminer l'expansion asymptotique de plusieurs racines imaginaires en utilisant une expansion en série de puissance.

L'un des deux outils de base de ce travail est le théorème de préparation de

Weierstrass car il permet de réduire la complexité de l'équation caractéristique à celle d'une équation polynomiale, suffisamment proche du point en question. Ledit polynôme a des propriétés adaptées à l'approximation prise. Par exemple, le polynôme a le même nombre de solutions que la multiplication de la racine en question. De plus, la forme et les propriétés de ce polynôme sont importantes, à tel point qu'il est possible d'utiliser la méthode du diagramme de Newton. Ainsi, de manière simple et graphique, les informations nécessaires pour déterminer le comportement pour déterminer le premier terme du développement sont obtenues à partir du polygone du diagramme. L'une des principales contributions est de coder la méthode en proposant un algorithme qui facilite son application, en évitant la représent- ation graphique.
D'autre part, il prend en compte les systèmes qui, sous certaines caractéristiques spécifiques du système, présentent une racine à comportement singulier (non borné). Le premier est un contrôleur PD standard (idéal) et le second est un contrôleur proportionnel basé sur le retard, où le retard apparaît via un opérateur de différence qui se rapproche de l'action dérivée. Après l'analyse asymptotique de la racine singulière, il y a des conditions dans lesquelles il y a un changement brusque du comportement qualitatif du système. En complément de l'analyse, le cas bien-posé est également considéré.
L'ingrédient principal de l'approche proposée est le calcul des solutions de la fonction caractéristique correspondante en tant que séries de puissance, de sorte que l'approche de stabilité par analyse asymptotique soit préservée tout au long du développement de la thèse.

Resumen extendido en español

Esta tesis se realizó a través de un trabajo de investigación científica que explora y analiza los efectos del retraso en la estabilidad de sistemas lineales. Se presta especial atención al comportamiento de las raíces críticas de la ecuación característica del sistema. Para hacer frente a la complejidad del problema, y debido a la presencia de un retraso como parámetro, se utiliza la continuidad de las raíces. Para hacer esto, se toma el espacio de parámetros determinado por el retraso, que se divide en regiones estables e inestables. Basado en la continuidad de las raíces; los cambios en la estabilidad se pueden determinar analizando la tendencia de las raíces en la frontera de estabilidad, utilizando el Teorema de la Función Implícita para calcular la tendencia a través de la frontera de estabilidad. Los objetivos esenciales discutidos en la tesis se presentan aquí brevemente. Descripción del comportamiento asintótico de varias raíces mediante un algoritmo, desdoblamiento de la caracterización de raíces múltiples mediante un solo criterio, y bajo un esquema de control particular, describir el comportamiento de raíces singulares. Así, la novedad de este trabajo es llenar estos vacíos y ampliar la metodología para utilizar una misma herramienta en diferentes situaciones, incluso en problemas mal-planteados. Es posible observar el movimiento de las raíces descrito por trayectorias continuas, que colapsan sobre el eje para formar una raíz múltiple. Es en el eje imaginario donde se concentra el estudio del comportamiento asintótico donde la multiplicidad es mayor que 1. Por tanto, en este caso, se pierde la diferenciabilidad de las raíces, lo que impide el uso del Teorema de la Función Implícita. Por este motivo, es necesario utilizar otra herramienta para el estudio de la multiplicidad. Para resolver este problema, se ha propuesto un procedimiento algebraico sistemático para determinar la expansión asintótica de varias raíces imaginarias utilizando una expansión en serie de potencias.

Una de las dos herramientas básicas de este trabajo es el Teorema de Preparación de Weierstrass, ya que permite reducir la complejidad de la ecuación característica

a la de una ecuación polinomial, suficientemente cercana al punto en cuestión. Dicho polinomio tiene propiedades adaptadas a la aproximación tomada. Por ejemplo, el polinomio tiene el mismo número de soluciones que la multiplicación de la raíz en cuestión. Además, la forma y las propiedades de este polinomio son importantes, tanto que es posible utilizar el Método del Diagrama de Newton. Así, de forma sencilla y gráfica, la información necesaria para determinar el comportamiento para determinar el primer término de la expansión se obtiene del polígono del diagrama. Uno de los principales aportes es codificar el método proponiendo un algoritmo que facilite su aplicación, evitando la representación gráfica.
Por otro lado, se tiene en cuenta los sistemas que, bajo determinadas características específicas del sistema, presentan una raíz con comportamiento singular (no acotado). El primero es un controlador de DP estándar (ideal) y el segundo es un controlador proporcional basado en retardo, donde el retardo aparece a través de un operador de diferencia que se aproxima a la acción derivada. Después del análisis asintótico de la raíz singular, hay condiciones bajo las cuales hay un cambio brusco en el comportamiento cualitativo del sistema. Además del análisis, también se considera el caso bien planteado.
El ingrediente principal del enfoque propuesto es el cálculo de las soluciones de la función característica correspondiente como series de potencias, de manera que el enfoque de estabilidad por análisis asintótico se conserva durante todo el desarrollo de la tesis.

This thesis is dedicated to my wife and to my parents. I am really grateful for the support and understanding throughout the entire doctorate.

Introduction

Time-Delay Systems

When describing the dynamical behavior of real-world processes, in a wide variety of fields and applications, Delay-Differential Equations (DDEs) provide an approach for capturing accurately their dynamics. Systems described by DDEs are also known as Time-Delay Systems (TDS), hereditary systems or systems with aftereffects. Recently such kinds of systems have gained a lot of attention since most of the influence and reactions occurring in engineering possess an inherent lag since these phenomena never occur instantaneously. In fact, such time-lags are mainly caused by the required time for the transportation of information, energy, or material.

We can trace back to the 18th century, to find the first model involving a delay-differential equation; first, in the works done by Euler and Bernoulli, then such equations appear in the works developed by Lagrange, Laplace Poisson, Concordet and others [18], but was until the early 20th century where the influence of the past states in the dynamical behavior of the system began to be taken into consideration. Moreover, in engineering, the impact of the delay phenomenon in the system's performance was observed until the decades of 40s, where DDEs gained the attention of scientists and engineers, as can be confirmed in [1].

Time-delay systems present challenges in the stability analysis, either in the computational approach or in the theoretical aspects. In particular, for the stability problem, this thesis is focused on the delay effect, as a varying parameter. For the problem of finding the characteristic roots on the stability boundary (the imaginary axis), it refers to [33, 72, 17] for the computation of such points. Using a continuity argument, the change in the stability properties can be traced by the asymptotic behavior of the characteristic roots within a delay interval.

Since this thesis is devoted to the analysis of the root behavior of Retarded Linear Time-Delay Systems, the following presents some preliminaries and a motivational example.

Linear Time-Invariant Delay Systems

In the last seven decades, the stability analysis of Linear Time-Invariant (LTI) system with time delay has been widely considered and great achievements have been made, see for instance [34, 33, 73, 85, 46, 91].
In this thesis, the focus is on retarded LTI systems with time delays described in the state-space form as:

$$\dot{x}(t) = A_0 x\left(t\right) + \sum_{k=1}^{q} A_k x\left(t - \tau_k\right), \qquad \tau_k \geq 0, \tag{1}$$

or by the differential-difference equation

$$y^{(n)}\left(t\right) + \sum_{\ell=0}^{n-1} \sum_{k=0}^{q} a_{k\ell} y^{(\ell)}\left(t - \tau_k\right) = 0, \qquad \tau \geq 0, \tag{2}$$

under appropriate initial conditions. In the above system equation, the system state is denoted by $x(t) \in \mathbb{R}$, the real constant matrices by $A_k \in \mathbb{R}^{r \times r}$ $(k = 0, \ldots, q)$ and the delay parameters as $\tau_k \in \overline{\mathbb{R}}_+$ $(0 < \tau_1 < \tau_2 < \cdots < \tau_q)$. Since the delays are point-wise, the delays τ_k have a selective effect, i. e., the state is shifted by a constant time period τ_k.
The initial condition is defined by

$$x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0],$$

where ϕ belong to the Banach space of continuous functions $\mathcal{C}\left([-\tau_q, \mathbb{R}^m]\right)$, see, for instance, [34]. Thus, the existence and uniqueness of the solutions is guaranteed for all initial conditions due to the linearity of

$$g(\phi) = A_0 \phi(0) - \sum_{i=1}^{q} a_i \phi(-\tau_i),$$

for further details, please, see [65].

The associated characteristic function is defined as follows.

Definition 0.0.1. *The function $f : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ given by*

$$f(s) = det\left(sI - A_0 - \sum_{i=1}^{q} A_i e^{-s\tau_i} \right), \tag{3}$$

is called the quasi-polynomial corresponding to the system (1)*, and the equation $f = 0$ is called the characteristic equation associated with the system.*

Remark 0.0.1. *The quasi-polynomial, defined above, is a transcendental function that has an infinite number of characteristic roots for $\tau_k > 0$, the reason why the system* (1) *represents a class of infinite-dimensional systems.*

Continuity of the Characteristic Roots

Stability of LTI delay systems is determined by the roots of the associated characteristic function. In addition, the nature of the characteristic roots play a vital role in stability analysis, in particular, the continuity is of utmost importance for this thesis.

First, the continuity of the roots when the characteristic function is defined by a polynomial is discussed to later continue with the dependency of the roots with the delay and the parameters in the case of quasi-polynomials.

It is well known that the roots of a monic polynomial depend continuously on the coefficients. This property can be formalized as follows [56].

Theorem 0.0.1. *Let*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = a_n \prod_{j=1}^{p} (z - z_j)^{m_j}, \quad a_n \neq 0,$$
$$Q(z) = a_n z^n + (a_{n-1} + \epsilon_{n-1}) z^{n-1} + \cdots + (a_0 + \epsilon_0)$$

and let

$$0 < r_k < \min |z_k - z_j|, \quad j = 1, 2, \ldots, k-1, k+1, \ldots, p.$$

There exists a positive number ϵ such that, if $|\epsilon_i| \leq \epsilon$ for $i = 0, 1, \cdots, n-1$, such that $Q(z)$ has precisely m_k roots in the disk D_k with center at z_k and radius r_k.

Roughly speaking, the continuity of the roots is due to the following arguments. Under appropriate choice of ϵ the polynomial $R(z) = \epsilon_{n-1}z^{n-1} + \cdots + \epsilon_1 z + \epsilon_0$ satisfies $|R(z)| < |P(z)|$, thus by Rouché's Theorem (see Appendix B), $P(z)$ and $Q(z)$ have the same number of roots in the disk D_k. With the exception of one point, the continuity of the polynomial roots is discussed in the following example.

Example 0.0.1. *Let $p(z)$ be a polynomial given by*

$$p(z) = a_1 z + a_0. \tag{4}$$

Now, suppose the pair (a_0, a_1) varies continuously across the path $T = T_1 \cup T_2$ from the initial point $P_0 = (\hat{a}_0, \hat{a}_1)$ as shown in the Figure 1. It's easy to see that

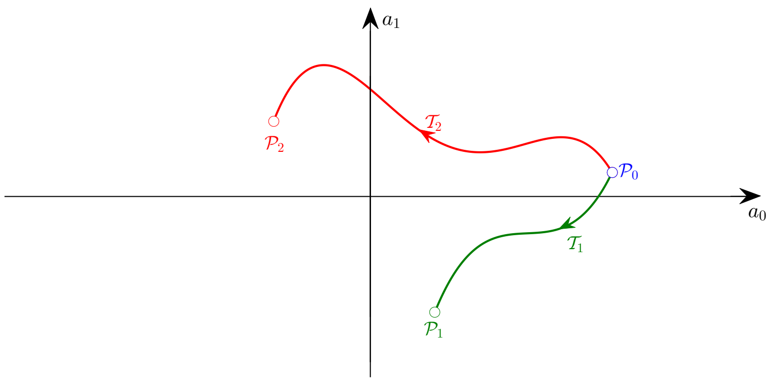


Figure 1: Parametric root path for the polynomial $p(z)$ (4).

$p(z)$ is Hurwitz for $\mathbf{a} = P_0$. Then, through the path T_2 towards the point P_2 the polynomial goes from stable to unstable. Moreover, by Theorem 0.0.1 the roots move in a continuous manner crossing the stability boundary. Now, if the pair (a_0, a_1) is varied through the path T_1 from P_0 to P_1, the stability of the polynomial also changes. When $a_1 = 0$, the polynomial loses degree which involves the presence of a root at infinity, as shown in Figure 2.

In the case of LTI delay systems, we keep track of the n polynomial roots and the remaining infinite many roots as τ is varied continuously. Furthermore, we pay special attention to critical roots located on the imaginary axis (see, for

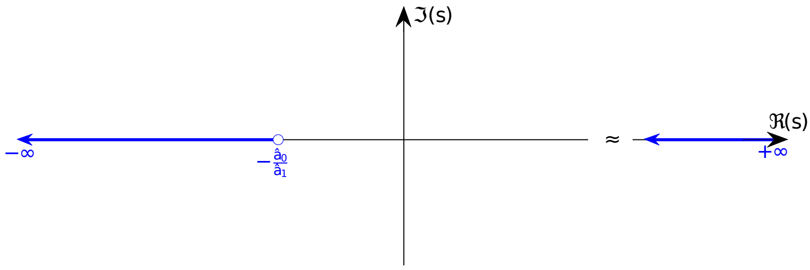


Figure 2: Root behavior of root at infinity of polynomial $p(z)$ (4).

instance, [25]). Extending the previous result concerning polynomials to quasi-polynomial and its variations with respect to system matrices and delays, the following proposition is presented (see [65]).

Proposition 0.0.1. *Let s_0 be a characteristic root of* (1) *with multiplicity m. There exist a constant $\tilde{\epsilon} > 0$ such that for all $\epsilon > 0$ satisfying $\epsilon < \tilde{\epsilon}$, there is a number $\delta > 0$ such that the quasi-polynomial*

$$f(s, \boldsymbol{\tau} + \delta\boldsymbol{\tau}, A_0 + \delta A_0, \ldots, A_q + \delta A_q)$$

where

$$\delta\boldsymbol{\tau} \in \mathbb{R}^q, \quad \|\delta\boldsymbol{\tau}\| < \delta, \ \boldsymbol{\tau} + \delta\boldsymbol{\tau} \geq 0$$
$$\delta A_k \in \mathbb{R}^{n \times n}, \quad \|\delta A_k\|_2 < \delta, \quad k = 0, \ldots, q,$$

has exactly m roots in the disk $\{s \in \mathbb{C} \colon |s - s_0| < \epsilon\}$.

Similar arguments, based on the Rouché's Theorem, to those used in the polynomial case allow proving the continuity of the quasi-polynomial roots.

Spectral Properties of the Quasi-Polynomial

Now, the focus is on the case of commensurate delay:

$$\tau_k = k\tau, \quad k = 1, \ldots, q,$$

where $\tau > 0$ showing the dependence of the delays. Let f in (3) be the corresponding quasi-polynomial which is explicitly given by

$$f(s,\tau) = \sum_{k=0}^{q} p_k(s) e^{-k\tau s}, \qquad \tau \geq 0, \tag{5}$$

where the polynomials p_k are given by

$$p_0(s) = s^n + \sum_{\ell=0}^{n-1} a_{0\ell} s^\ell, \quad p_k(s) = \sum_{\ell=0}^{n-1} a_{k\ell} s^\ell, \quad k = 1, \ldots, q.$$

In this manner, the stability analysis is reduced to the locations of the roots of the quasi-polynomial f (5). The following statements are of the utmost importance to stability analysis of LTI delay systems (see, for instance, [65, 27, 34]).

(i) For $\tau = 0$ f is reduced to a polynomial with $\operatorname{card}(\sigma(f)) = n$;

(ii) an infinite number of roots appear in $\mathbb{C}_-$, for $\tau > 0$;

(iii) the root path can be traced back continuously with respect to τ.

With respect to the stability of the system; based on a condition on the set of characteristic roots, the spectrum of f (denoted by $\sigma(f)$), the following definition is given.

Definition 0.0.2. *The characteristic function given by the quasi-polynomial* (3) *associated to the LTI system* (1), *is called stable if the following conditions hold*

$$\{s \in \mathbb{C} \colon \Re(s) \geq 0, \ s \in \sigma(f)\} = \varnothing.$$

In the following paragraphs, the qualitative behavior of the critical roots will be discussed in greater detail. Generally speaking, the approach is as follows. First, it is assumed that the system is stable when $\tau = 0$ and it is assumed that there are two critical delays values. Consider the delay interval of interest $\Omega := [0, \tau^u]$, and denote by τ_k^* the critical values such that

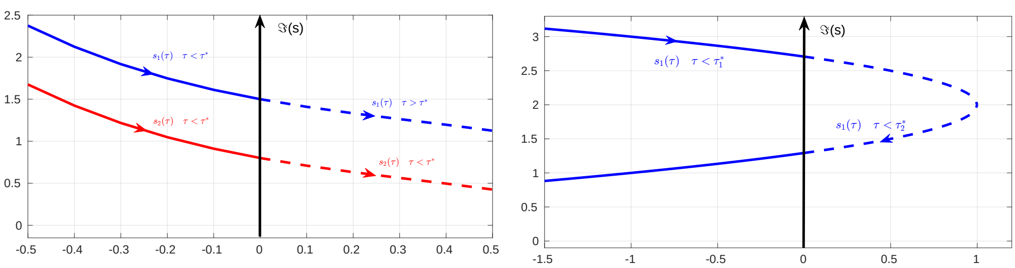
$$\tau_1^* < \tau_2^*, \quad \tau_k \in \Omega.$$

By a continuity argument on the root (see **??** for a discussion on the subject), the roots must intersect by the imaginary axis.

From the above description it can inferred that the system is stable for $0 \leq \tau \leq \tau_1^*$. However, the stability for $\tau \geq \tau_1$ remains unknown. Now the focus is on the root behavior around the critical value by considering some graphical examples. In other words, the way in which the roots intersect the imaginary axis (the boundary-crossing) when the delay values reach $\tau = \tau_1^*$ and τ_2^* for $\tau \in \Omega$ will be investigated. The following examples of the local behavior of the critical roots focus on simple roots, multiple roots will be treated in Chapter 2.

The Figure 3 gives more details of this behavior when τ varies inside Ω. In order to illustrate this phenomenon, two possible cases are given. Since, with respect to the real axis, the root locus is symmetric, we only consider the upper half of the complex plane. Figure 3-(a) shows the root behavior with two stable roots for $0 \leq \tau \leq \tau_1^*$ and changing to instability for $\tau_1^* < \tau \leq \tau^u$. If a root behaves as in Fig. 3-(b), the root is stable for $0 \leq \tau \leq \tau_1^*$, then changes to an unstable root for $\tau_1^* \leq \tau \leq \tau_2^*$ and return to stability for $\tau_2^* \leq \tau \leq \tau^u$. More on this analysis can be found [82].

In the previous discussion, the crossing of the critical roots $\boldsymbol{i}\omega^*$ at the critical



(a) Crossing of two simple roots (b) Crossing of a particular simple root

Figure 3: Smooth trajectories when crossing the imaginary axis.

delay τ^* was considered. In some situations, it is possible to have non-regular or degenerate cases. This behavior can present some subtleties that cannot be studied with conventional techniques. To see this, let's take a look at Fig. 4, the root crosses the imaginary axis inducing instability but the root path loses its smoothness. Now let's move to Fig. 5; if the root locus is of the form shown in

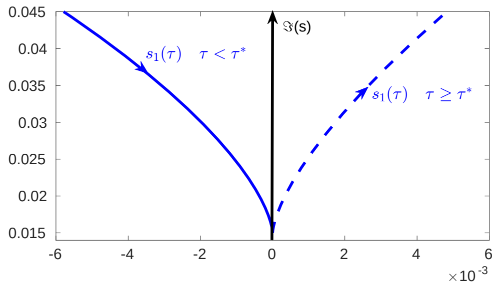
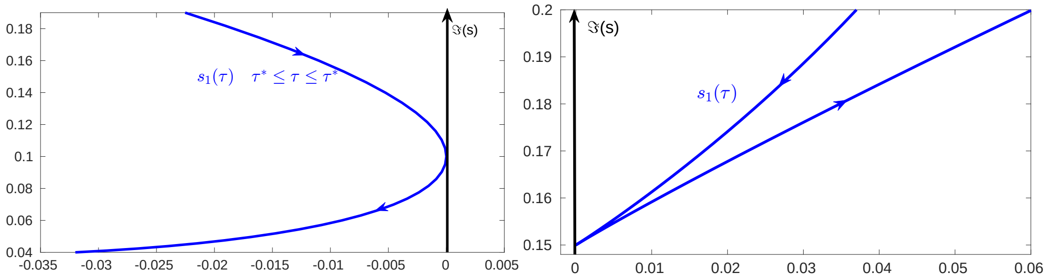


Figure 4: Smoothness of the root lost on the imaginary axis.

Fig. 5-(a), the root is stable throughout the delay interval Ω. In the case of Fig. 5-(b), the root locus loses smoothness in the form of a cusp. The system remains unstable in the delay interval Ω since one root stays on the left-hand side. Figure 5 gives more details of this behavior. From the discussion above it appears that the



(a) Simple root remaining stable

(b) Multiple root (non semisimple) remaining in instability: degenerate case

Figure 5: Roots locus examples.

asymptotic analysis problem may present a special situation for the study of the local root behavior. Such an analysis is the main objective of thesis work for LTI delay systems.

Stability Analysis of LTI Delay Systems

The observations considered in the previous section have been deeply explored in [24], where, for a general *retarded* linear time-invariant delay system with commensurate delays, the authors have first, fully characterized the stability properties of such a system by finding a set of critical delay values, at which the system's characteristic quasi-polynomial has critical zeros on the imaginary axis. Secondly, considering the delay as a *variable parameter* and by adopting an operator based-approach they have expanded the solutions of the quasi-polynomial in terms of a Taylor (or Puiseux) series, allowing analyzing the behavior of the solutions as the delay varies around a critical delay value. As discussed in [23, 66], even in the case of a fixed delay, the testing of stability for a time-delay system is not a simple task. To introduce a simple example of the stability analysis of DDE, let's consider a mechanical system where the feedback is proportional delayed state values. This problem was first solved in [39] and solve this problem using stability charts of the parameters in [40].

Example 0.0.2. *The system of interest is the damped delayed oscillator described by the scalar delay differential equation:*

$$\ddot{x}(t) + \alpha\dot{x}(t) + \beta x(t) = \eta x(t - \tau), \tag{6}$$

where α, β, η *are control parameters and* $\tau \in \mathbb{R}_+$ *is the time-delay parameter.*
Since the oscillator described by (6) *is stable if all the roots of its characteristic equation have a negative real part, the complete stability analysis requires the determination of the triplet* $(\alpha, \beta, \eta) \in \mathbb{R}^3$ *and the time-delay* $\tau \in \mathbb{R}_+$ *which is assumed to be constant. Multiple roots are also of great importance since can lead to complex behaviors. For instance, in [16] the dominancy of the maximal multiple roots on second-order system was treated. In the case of* (6) *the multiplicity of any given root is bounded by* 4*. More on the analysis of the behavior of multiple roots will be given in Chapter 2.*
First, the problem is reduced by considering the special case when $\eta \equiv 0$*, i.e., the* damped oscillator *with characteristic equation:*

$$P(s) := s^2 + \alpha s + \beta = 0.$$

Simple computation showed that the nature of the characteristic roots, which are given by two complex roots as follows

$$s_{1,2} = -\frac{\alpha}{2} \pm \frac{1}{2}\left(\alpha^2 - 4\beta\right)^{1/2}.$$

Table 2: Description of D-curves

Oscillator	D-curve	Description
Damped	$\omega = 0\colon \alpha = 0,\ \beta \in \mathbb{R}$ $\omega \neq 0\colon \beta = 0,\ \alpha = \omega^2$	straight line
Undamped delayed	$\omega\tau \neq n\pi\colon \eta = 0,\ \beta = \omega^2$ $\omega\tau = n\pi\colon \eta = (-1)^n\beta + (-1)^{n+1}\left(\frac{n\pi}{\tau}\right)^2$	45-deg lines
Damped and delayed	$\omega = 0\colon \eta = \beta$ $\omega\tau \neq n\pi\colon \beta = \omega^2 + \frac{\alpha\omega cos(\omega\tau)}{sin(\omega\tau)},\ \eta = \frac{-\alpha\omega}{sin(\omega\tau)}$	Surfaces

Thus, the stability chart can be determined in the parameter plane (β, α) *depicted in Fig. 6. This chart is defined by by* $P(\boldsymbol{i}\omega) = -\omega^2 + \alpha\boldsymbol{i}\omega + \beta$*, with* $\omega > 0$*, summarized in table 2. Hence, such a stability chart can find a stability region in the parameter space* (β, α)*. Next, in order to gain insights into the complete problem, the damping term* α *is set to zero and the* undamped delayed oscillator *rewrites as:*

$$\ddot{x}(t) + \beta x(t) = \eta x(t - \tau).$$

In this case, the associated quasi-polynomial is given by

$$P(s) := s^2 + \beta - \eta e^{-s\tau}. \tag{7}$$

Now, with the aim of finding the complete parameter set β, η *for which there exists at least a solution on the imaginary axis* $s = \boldsymbol{i}\omega$ *is considered. Thus by equation the real and imaginary part to zero, we find:*

$$-\omega^2 + \beta - \eta cos(\omega\tau) = 0,$$
$$\eta sin(\omega\tau) = 0.$$

From the equations above the curves depicted in Fig. 7 (blue lines) are obtained. The condition $\omega\tau \neq n\pi$ *represents a horizontal line that starts from the coordinate* $(\omega^2, 0)$*. If the condition* $\omega\tau = n\pi$ *and fixed delay* τ *are assumed the system* (6) *gives a set of straight lines with slope* $(-1)^k$ *intersecting the* η *axis at* $k^2/4$*. Thus, the stability chart is characterized by triangular regions.*

When comparing the two perpendicular lines in the parameters space (β, α) *with*

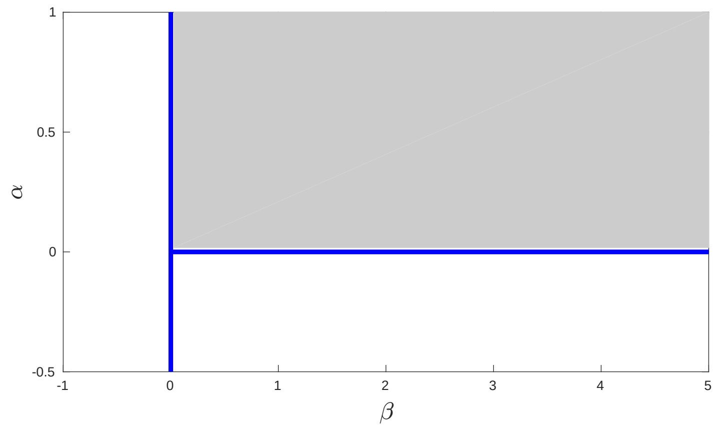


Figure 6: Stability Charts of Damped Oscillator. In the gray region there is no unstable roots.

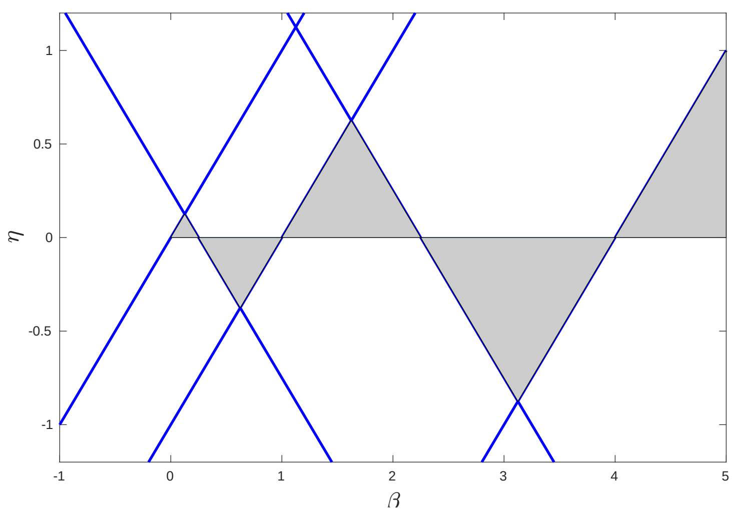


Figure 7: Stability Charts of Damped Oscillator. In the gray region there is no unstable roots.

the formation of triangles in (β, η)*, see Figure 7, we obtain insights into the complete subdivision of the parameter space* (α, β, η) *for the damped delayed oscillator.*

It is possible to obtain the crossing direction *by considering the tendency of the root with respect to* η *of the quasi-polynomial in* (7)*. To this end, the implicit function theorem is used to compute the derivative of* σ *with respect to* η *and the condition* $\omega\tau = n\pi$ *applied:*

$$\frac{d\sigma}{d\eta} = \frac{\eta\tau}{\eta^2\tau^2 + 4\omega^2}.$$

If the condition $\omega\tau \neq n\pi$*, on the d-curves the crossing direction is given by*

$$\frac{d\sigma}{d\eta} = \frac{-sin(\omega\tau)}{2\omega}.$$

By means of this approach, we can derive the number of unstable roots in each region.

Now, if the damped and delayed parameters are $\alpha \neq 0$ *and* $\eta \neq 0$*, then the damped and delayed oscillator is considered. Stability is guaranteed by the roots of the associated quasi-polynomial:*

$$P(s, \tau) = \left[s^2 + \alpha s + \beta\right] - \eta e^{-s\tau}. \tag{8}$$

Setting $s = \boldsymbol{i}\omega$ *and taking the real and complex part of the curves defined by* $P(\boldsymbol{i}\omega, \tau)$ *are given in Table 2, that belong to the parameter space* (α, β, η)*. In order to understand this, based on the continuous dependence on the parameters, the undamped delayed oscillator is taken as base and varying* α *from zero. This behavior is shown in Figure, as the projection of the surface to the plane* (β, η) *when* $\alpha = 0.1$ *and fixed delay* $\tau = 2\pi$*.*

Remark 0.0.2. *As discussed in* **??** *, the continuity of the roots is of key importance for stability analysis. Even though the characteristic roots are continuous functions of the parameters, either polynomial or quasi-polynomial, at multiple roots the differentiability is lost.*

Other second-order systems can be found in the literature. For instance, the effect of delayed feedback on the limit cycle considered in the Van der Pol's oscillator [8].

Multiple Roots

In order to perform the testing of stability for time-delay systems with multiple roots, and due to the loss of continuity, the root behavior should be analyzed in detail. For instance, in [43], the double root is studied, and the splitting behavior of multiple critical roots has also been considered by [51].

Definition 0.0.3. *The points y_0 at x_0 of the equation $f(x_0, y) = 0$ satisfying*

$$f(x_0, y_0) = \frac{\partial f}{\partial y} = \cdots = \frac{\partial^{m-1} f}{\partial y^{m-1}} = 0, \; \frac{\partial^m f}{\partial y^m} \neq 0,$$

is called a multiple root of multiplicity m.

From the previous definition, it can be seen that the Implicit Function Theorem cannot be applied, which leads to approach the problem of multiple roots from a different perspective. Instead of analyzing the stability behavior of a time-delay through their corresponding quasi-polynomial, such an analysis is performed through a polynomial (with a degree equal to the multiplicity of the critical zero) that preserves the full information concerning the stability properties.

Delay-dependent Parameters

In [12], the authors present some consideration for the analysis of quasi-polynomials with delay-dependent coefficients:

$$f(s, \tau) = p_0(s, \tau) + p_1(s, \tau)e^{-s\tau}.$$

The coefficients are continuous and differentiable functions given by

$$p_0(s, \tau) = \sum_{k=0}^{n} p_{0,k}(\tau)s^k, \qquad p_1(s, \tau) = \sum_{k=0}^{m} p_{1,k}(\tau)s^k, \quad n > m,$$

such that $p_{0,k}, p_{1,k} : \mathbb{R}_+ \to \mathbb{R}$. In order to ensure a well-behaved polynomial, the following *assumptions* were made:

(1) $p_0(\boldsymbol{i}\omega, \tau) + p_1(\boldsymbol{i}\omega, \tau) \neq 0, \; \forall(\omega, \tau) \in \mathbb{R}^2_{\geq 0}$;

(2) $\limsup \left\{ \left| \frac{p_1(s,\tau)}{p_0(s,\tau)} \right| : |s| \to \infty, \Re(s) \geq 0 \right\} < 1$ for any τ;

(3) $F(\omega, \tau) := |p_0(\boldsymbol{i}\omega, \tau)|^2 - |p_1(\boldsymbol{i}\omega, \tau)|^2$, for each τ has at most a finite number of real roots;

(4) each positive root $\omega(\tau)$ of $F = 0$ is continuous and differentiable.

In Chapter 4, a particular case will be analyzed in such a way that an explicit expression for the non-bounded roots for small delay values will be given. To the best of the author's knowledge, such an analysis was not presented in the open literature and the proposed results represent a novelty.

Problem Statement

Based on the above discussions, one can conclude that the use of delay as a design parameter can positively impact system performance providing useful properties such as stability, robustness, or noise attenuation. Previous paragraphs open the following problems:

1 Describe the asymptotic behavior of the roots around a critical parameter.

2 Extend the work to two delay parameters.

3 Analyze the case of non-bounded characteristic roots of the closed-loop of some SISO systems in two configuration schemes.

Objective 1. The asymptotic behavior of multiple roots, compared with the behavior of simple roots, displays a more complex situation. In this vein, it will be of core importance to describe its analytical and geometrical properties around a critical parameter.

Objective 2. Characterize the root behavior around a critical solution by considering the interaction of two delays. The method to approach stability analysis within which this work is contained can be applied to the case of two parameters. It is worth mentioning that, unlike the case of a single parameter, in the multi-parameter case, there exist some singular and unexpected behaviors, which must be taken into account.

Objective 3. The analysis will be focused on the root behavior of the associated quasi-polynomial of a closed-loop system subject to a PD-control when a delay-difference operator is used to approximate the derivative action. In particular, in the case when the corresponding stability problem is *ill-posed* for "small" delay values.

For future work, in Section 4.8, the case of LTI Delay Systems with delay-dependent parameters is presented as an extension of Objective 3. Then, an extension of Objectives 1 and 2 is proposed. Thus, the extension to the analysis of multiple roots under variations of more than two parameters is motivated.

Thesis Outline

The remaining part of the thesis is organized as follows.
In Chapter 1, the mathematical tools used in the thesis are presented, as well as some useful definitions and notations. In particular, the Weierstrass Preparation Theorem is presented as a local form around multiple roots. For the asymptotic analysis, fractional power series are defined, along with the main definitions. The Weierstrass polynomial is used for the asymptotic analysis of multiple critical roots of quasi-polynomials, Chapter 2. This analysis is carried out both numerically and analytically. As a main result, a simple criterion that allows us to describe the splitting of the roots is given.
The approach to compute splitting of multiple is extended in Chapter 3, by considering two non-commensurate delays as variable parameters. In addition, the considerations for which construction is possible are shown.
The ill-posed situation of the approximation of the derivative action will be analyzed in Chapter 4. In particular, the roots at infinity will be appropriate by means of the computation of the first terms of the series expansion.
The thesis closes with the presentation of conclusions and prospects for future work in Section 4.8.

Chapter 1

Definitions and Prerequisites

In this chapter, the basic framework on which the thesis is based is introduced, starting with some general definitions. The aim of this chapter is twofold; first, give a local representation of the characteristic function around the desired point through the use of the so-called *Weierstrass Polynomial.* Secondly, introduce the Newton Diagram Method as the main tool for the computation of roots.

1.1 General Definitions

It is well known that the ring of polynomials is denoted by $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_n]$. Furthermore, polynomials can be added and multiplied in the usual way and from the choice of complex coefficients can be factorized. As an extension we allow the variable x_i to have negative exponents, defining the ring of Laurent polynomials $\mathbb{C}[\mathbf{x}]$. As a generalization of a polynomial (see, for instance, [67]), with a finite number of terms, a *formal power series* is an expression with infinitely many terms of the form:

$$\sum_{i=0}^{\infty} a_i x^i, \quad a_i \in \mathbb{C},$$

which can also be added and multiplied in the usual way to form a ring denoted by $\mathbb{C}[[\mathbf{x}]]$. By requiring convergence, we denote by $\mathbb{C}\{\mathbf{x}\}$ the field of *convergent power series.*

The *Puiseux series* can be defined as a generalization of power series, which allow fractional exponents with bounded denominator and even negative exponents. Let $K_{x,d} = \mathbb{C}\{x^{1/d}\}$ be the field of *Puiseux series* with complex coefficients. Thus, it is closed under the operation of addition, multiplication, and division, from its field properties.

Definition 1.1.1. *The formal power series $f(x) \in \mathbb{K}$:*

$$f(x) = \sum_{i=i_0}^{\infty} a_i x^{i/d} \quad i_0 \in \mathbb{Z},\ d \in \mathbb{N},$$

with coefficients $a_i \in \mathbb{C}$, is denoted as Puiseux series.

From the above, we can see that Puiseux series $f(x)$ can also be defined as Laurent power series in $x^{1/m}$. Therefore, these series can be added and multiplied as if they were polynomials, taking into account the following definition.

Definition 1.1.2. *Let $f(x_1, x_2) = \sum_{i,j} a_{i,j} x_1^i x_2^j$ be a formal power series. The* order ord (f) *is the smallest number $n = i + j$ such that $a_{i,j} \neq 0$.*

Hence, for two formal power series $\phi(x)$, $\psi(x)$,the order is a function ord: $\mathbb{C}\,[[\mathrm{x}]] \to \mathbb{Q}$ with the following properties:

1. $\operatorname{ord}(\phi) = \infty$ if and only if $\phi = 0$,
2. $\operatorname{ord}(\phi + \psi) \geq \min\{\operatorname{ord}(\phi), \operatorname{ord}(\psi)\}$,
3. $\operatorname{ord}(\phi\psi) = \operatorname{ord}(\phi) + \operatorname{ord}(\psi)$.

1.2 Algebraic Functions

The spectral properties of the associated quasi-polynomial are of fundamental importance in the stability analysis of LTI delays systems. In particular, the smoothness properties of its roots possess a leading role in the stability analysis methods. For this reason, we also consider the non-regular cases of the spectra. For simplicity let's consider the following bivariate polynomial:

$$p(z, x) = a_n(x) z^n + a_{n-1}(x) z^{n-1} + \cdots + a_0(x) \tag{1.1}$$

where the coefficient $a_i(x) \in \mathbb{C}[x]$. Next, the definition of an algebraic function is given, see for instance [14].

Definition 1.2.1. *An Algebraic Function is a function $z = \phi(x)$ with $\phi \colon \mathbb{C} \to \mathbb{C}$ such that*

$$p\left(\phi(x), x\right) = 0.$$

The main objective is the characterization and the description of the roots, in particular, the root behavior with respect to parameter variations, for these reasons the *discriminant* is taken as a tool in this study. The discriminant of p is a function that depends on the coefficients and can be defined, up to a constant factor, as the product of the square of its roots, (see, for instance, [90]).

Definition 1.2.2. *Let $z_i = \phi_i(x)$ be m-algebraic functions of the polynomial $p(z, x)$* (1.1)*. Then, the discriminant Δ is defined as:*

$$\Delta(\mathbf{x}) := \prod_{i<j} \left(\phi_i(x) - \phi_j(x)\right).$$

In general, it is also possible to write the discriminant in terms of the polynomial coefficients. For higher-order degree polynomials, the discriminant can also be defined using the *resultant*. In this regard, the resultant of two polynomials f and g is defined as the determinant of a particular matrix whose coefficients are composed of those of f and g. More precisely, let $f \in \mathbb{C}[x]$ and $g \in \mathbb{C}[x]$ be given by

$$f(x) := \sum_{j=0}^{n} a_j x^j, \qquad g(x) := \sum_{j=0}^{m} b_j x^j,$$

where $a_0 \neq 0$ and $b_0 \neq 0$. The *resultant* of f and g (see, for instance, [76]), is denoted by $\mathcal{R}\left(f, g\right) \in \mathbb{C}[x]^{d \times d}$, where $d = m + n$, and defined as the determinant of the following matrix:

$$\mathcal{R}\left(f, g\right) := \det \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & \cdots & a_0 & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & a_n & a_{n-1} & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & & & \\ & b_m & b_{m-1} & \cdots & \cdots & b_0 & & \\ & & \ddots & \ddots & \ddots & & \ddots & \\ & & & b_m & b_{m-1} & \cdots & \cdots & b_0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right\} n \text{ rows} \end{matrix}$$

An interesting property of the resultant is that $\mathcal{R}(f,g)=0$ if and only if f and g have a common zero. As an example, the resultant of the quadratic polynomial $p(z,\mathbf{x})=a_2(\mathbf{x})z^2+a_1(\mathbf{x})z+a_0(\mathbf{x})$ and its derivative $p'(z,\mathbf{x})=2a_2(\mathbf{x})z+a_1(\mathbf{x})$ is given by

$$\mathcal{R}(p,p')=\det\begin{bmatrix} a_2(\mathbf{x}) & a_1(\mathbf{x}) & a_0(\mathbf{x}) \\ 2a_2(\mathbf{x}) & a_1(\mathbf{x}) & 0 \\ 0 & 2a_2(\mathbf{x}) & a_1(\mathbf{x}). \end{bmatrix} = -a_2(\mathbf{x})[a_1(\mathbf{x})^2-4a_2(\mathbf{x})a_0(\mathbf{x})].$$

Now, the discriminant of a polynomial p of degree n is related by the resultant as follows

$$\Delta(\mathbf{x})=\frac{(-1)^{\left(\frac{n(n-1)}{2}\right)}}{a_n(\mathbf{x})}\mathcal{R}(p,p'),$$

hence, the discriminant of the quadratic polynomial is given by $\Delta(\mathbf{x})=a_1^2(\mathbf{x})-4a_2(\mathbf{x})a_0(\mathbf{x})$.

For a polynomial of the form (1.1) it is generally assumed that the following conditions hold.

i) The leading coefficient satisfies $a_n\neq 0\ \forall\mathbf{x}$;

ii) $a_n,\cdots,a_0$ have no common factor involving $\mathbf{x}$;

iii) $\Delta(p)\not\equiv 0$.

A classification of the set of polynomial roots $\{(z,\mathbf{x})\colon p(z,\mathbf{x})=0\}$ is often found in literature (see, for instance, [14]). The above assumptions do not represent serious constraints. For instance, if $\Delta(p)$ is identically zero, the elimination of all except one of these repeated factors reduce the polynomial to one that satisfies iii). Further implications of these conditions lead to the following definition.

Definition 1.2.3. *A point $\mathbf{x}_0\in\mathbb{C}^r$ is called an ordinary point of the algebraic function $z=\phi(\mathbf{x})$ if the coefficient $a_n(\mathbf{x}_0)\neq 0$ and the discriminant $\Delta_p(\mathbf{x}_0)\neq 0$. If one or both $a_n(\mathbf{x}_0)=0$, $\Delta_p(\mathbf{x}_0)=0$, x_0 is a singular point.*

In the above root classification, regular roots represent *simple roots*, which can be defined by the partial derivatives

$$p(z_0,\mathbf{x}_0)=0,\qquad \left.\frac{\partial p(z,\mathbf{x})}{\partial z}\right|_{(z_0,\mathbf{x}_0)}\neq 0,$$

where z_0 is the value of the algebraic function at $\mathbf{x}_0$. A singular point can be either multiple roots

$$p(z_0, \mathbf{x}_0) = \left.\frac{\partial p(z, \mathbf{x})}{\partial z}\right|_{(z, \mathbf{x}_0)} = \cdots = \left.\frac{\partial^{m-1} p(z, \mathbf{x})}{\partial z^{m-1}}\right|_{(z, \mathbf{x}_0)} = 0, \quad \left.\frac{\partial^{m} p(z, \mathbf{x})}{\partial z^{m}}\right|_{(z, \mathbf{x}_0)} \neq 0,$$

a point where $a_n(\mathbf{x}_0) = 0$, or both of $\Delta(\mathbf{x}_0)$, $a_0(\mathbf{x}_0)$ vanish. The point at infinity, $\mathbf{x}_0 = \infty$ is also considered as a singular point. This phenomenon was observed in Example 0.0.1 where the root crosses the boundary at infinity when the polynomial loses degree. According to its singular behavior, this classification can be extended to the *solutions* of quasi-polynomials as will be discussed in Section 4.8.

1.3 Local Form of Holomorphic Functions

The approach to the analysis of spectral behavior is based on the local properties of the roots of the characteristic function. Since the main motivation of this thesis is the asymptotic behavior of the roots of quasi-polynomials as functions of the time-delay, we discuss the Implicit Function Theorem and the Weierstrass Preparation Theorem here in brief.

1.3.1 Implicit Function Theorem

The *Implicit Function Theorem* gives conditions under which it is possible to solve for z as a function of x in the neighborhood of a known solution and is used as a basic tool in the asymptotic analysis.

Definition 1.3.1. *An equation of the form*

$$f(z, \mathbf{x}) = c$$

implicitly defines z as a function of $\mathbf{x}$ on a domain $V \subset \mathbb{C}$ if there is a function ϕ on V for which $f(\phi(\mathbf{x}), \mathbf{x}) = c$ holds for all $\mathbf{x} \in V$.

The above equation can be simplified assuming that $c = 0$, given the traditional form $f = 0$. A standard tool in the study of differentiable functions is the *Implicit Function Theorem.*

Theorem 1.3.1 (Implicit Function Theorem,[37]). *Let* $f_j(\mathbf{z},\mathbf{x})$*,* $j = 1,\ldots,m$*, be analytic functions of* $(\mathbf{z},\mathbf{x}) = (z_1,\ldots,z_m,x_1,\ldots,x_n)$ *in a neighborhood of a point* $(\mathbf{z}_0,\mathbf{x}_0)$*,* $U_{\mathbf{z}_0}\times V_{\mathbf{x}_0}\subset\mathbb{C}^m\times\mathbb{C}^n$*, and assume that* $f_j(\mathbf{z}_0,\mathbf{x}_0) = 0$*,* $j = 1,\ldots,m$ *and that*

$$\det\left(\frac{\partial f_j}{\partial z_k}\right)_{j,k=1}^{m} \neq 0 \quad \text{at } (\mathbf{z}_0,\mathbf{x}_0)\,.$$

Then the equations $f_j(z,\mathbf{x}) = 0$*,* $j = 1,\cdots,m$*, have a uniquely determined analytic solution* $\mathbf{z} = \phi(\mathbf{x})$ *in a neighborhood of* $\mathbf{x}^0$*, such that* $\phi(\mathbf{x}_0) = \mathbf{z}_0$*.*

There are several implicit function theorems, each of these theorems made different assumptions arriving at different conclusions. In this vein, Table 1.1 gives a summary of these hypothesis and conclusions. Thus, under some conditions, ϕ over its partial derivatives can be computed in a simple manner using f, since its uniquely defined and continuous.

Table 1.1: Guide to Implicit Function Theorem

Hypothesis	Conclusions
f is continuous, and $D_{\mathbf{x}}f(\mathbf{z}_0,\mathbf{x}_0)$ is invertible	$f(\phi(\mathbf{x}),\mathbf{x}) = f(\mathbf{z}_0,\mathbf{x}_0)\ \forall x\in U$, such that $\phi(\mathbf{x}_0) = \mathbf{z}_0$
$D_{\mathbf{x}}f$ is continuous	ϕ is unique in U, and continuous on V
f is C^k on U	ϕ is unique in U, and C^k on V

Example 1.3.1. *We consider as an example the unit circle in* $\mathbb{R}^2$*, which can be described as*

$$\left\{(x_1,x_2)\in\mathbb{R}^2 | x_1^2 + x_2^2 = 1\right\}.$$

Alternatively, we can also write the unit circle as $f(x_1,x_2) = x_1^2 + x_2^2 - 1$*, with differential:*

$$Df(x_1,x_2) = (2x_1\ 2x_2),$$

suggesting the use of Implicit Function Theorem, with $x_2 = \pm\phi(x_1)$*. To represent the unit circle, we could use the following graph patches:*

$$x_2 = \begin{cases} (1-x_1^2)^{1/2}, & x_1\in(-1,1) \\ -(1-x_1^2)^{1/2}, & x_1\in(-1,1). \end{cases}$$

Next, we give the generalization for the local behavior of multiple roots.

1.3.2 Weierstrass Preparation Theorem

In this section, we consider the situation where the point of interest $(z, \mathbf{x}_0)$ is multiple roots. We will present the main tool for the analysis of multiple roots which allows us to guarantee the existence of convergent power series solutions of the equation $f(z, \mathbf{x}) = 0$, assuming f is a holomorphic function.
It is possible to reduce the analytic properties of $f(z, \mathbf{x})$ to algebraic properties. To this purpose, let us consider the following result.

Theorem 1.3.2 (Weierstrass Preparation Theorem [55]). *Let $f(z, \mathbf{x})$ be an analytic function vanishing at the singular point $z_0 \in \mathbb{C}$, $\mathbf{x}_0 \in \mathbb{C}^n$, where $z = z_0$ is an m-multiple root of the equation $f(z, \mathbf{x}) = 0$, i.e.,*

$$f(z_0, \mathbf{x}_0) = \left.\frac{\partial f}{\partial z}\right|_{(z_0, \mathbf{x}_0)} = \cdots = \left.\frac{\partial^{m-1} f}{\partial z^{m-1}}\right|_{(z_0, \mathbf{x}_0)} = 0, \quad \left.\frac{\partial^m f}{\partial z^m}\right|_{(z_0, \mathbf{x}_0)} \neq 0.$$

Then, there exist a neighborhood $U_0 \subset \mathbb{C}^{n+1}$ of the point $(z_0, \mathbf{x}_0) \in \mathbb{C}^{n+1}$ in which the function $f(z, \mathbf{x})$ can be expressed as

$$f(z, \mathbf{x}) = W(z, \mathbf{x})\, b(z, \mathbf{x}), \tag{1.2}$$

where

$$W(z, \mathbf{x}) = (z - z_0)^m + w_{m-1}(\mathbf{x})(z - z_0)^{m-1} + \cdots + w_0(\mathbf{x})$$

and $w_0(\mathbf{x})$,...,$w_{m-1}(\mathbf{x})$, $b(z, \mathbf{x})$ are analytic functions uniquely defined by the function $f(z, \mathbf{x})$ and $w_i(\mathbf{x}_0) = 0$, $b(z_0, \mathbf{x}_0) \neq 0$.

For proof and discussion on this theorem see, for instance [19, 45, 90]. The nature of the analytic function $W(z, \mathbf{x})$ allows the study of the zeros of F from an algebraic perspective, reducing the complexity of the problem.

Remark 1.3.1. *It can be seen from Theorem 1.3.2 that since $b(z, \mathbf{x})$ is an holomorphic non vanishing function at $(0, \mathbf{0})$, then there must exist some neighborhood $\Omega(0, \mathbf{0}) \subset \mathbb{C}^{n+1}$ at which $b(z, \mathbf{x})$ preserves the same property. Hence, based on this observation, we can ensure that the root-locus of a given quasi- polynomial f in the neighborhood Ω will be the same as the root-locus of $W(z, \mathbf{x})$.*

Implicit Function Theorem 1.3.1 along with Weierstrass Preparation Theorem 1.3.2 are our main tools for dealing with implicit functions; as a consequence of this, we present the following Corollary 1.3.1.

Corollary 1.3.1. *Let f be an analytic function, and assume f vanishes at the origin. Then, the set of zeros of f in a neighborhood of the origin is of dimension $n-1$.*

Example 1.3.2. *In order to clarify the advantages of the Weierstrass Preparation Theorem let us consider an example borrowed from [80]. Let polynomial*

$$f(z, x_1, x_2) = z^4 + (-1 + x_2 + x_1^2)z^3 + (-1 + x_1x_2)z^2 + (1 - 2x_1)z + x_1 + x_2^2$$

posses a double root $z_0 = 1$ at $(x_1, x_2) = (0, 0)$. Then, the local form is given by the Weierstrass polynomial $W(z, \mathbf{x})$ as follows

$$(z-1)^2 + w_1(\mathbf{x})z + w_0(\mathbf{x}),$$

with coefficients given by convergent power series:

$$\begin{aligned} w_1(\mathbf{x}) &= \frac{1}{4}(-x_1 + 3x_2) + \frac{1}{16}\left(8x_1^2 + 3x_1x_2 - 13x_2^2\right) + o\left(|x_1||x_2|\right) \\ w_0(\mathbf{x}) &= \frac{1}{2}(-x_1 + x_2) + \frac{1}{16}\left(13x_1^2 + 4x_1x_2 + 7x_2^2\right) + o\left(|x_1||x_2|\right). \end{aligned}$$

In Chapter 2 two methods are given for the computation of of the coefficients $w_i(\mathbf{x})$. The splitting of the double root at z_0 and the computation of the coefficients $w_1(x_1, x_2)$ and $w_1(x_1, x_2)$ will be deeply explored in Chapter 2.

The purpose of the next section is to provide a tool for computing the power series solutions around multiple roots.

1.3.3 Quasi-ordinary Roots

It is well known that the perturbation effect of one parameter on multiple roots can be expressed employing the Puiseux series. This process can be extended to the multi-parameter perturbation utilizing the so-called Abhyankar-Jung Theorem (see, for instance, [2]). We introduce the following concept of quasi-ordinary polynomials.

Definition 1.3.2. *Let* $f(z, x_1, x_2)$ *be a Weierstrass polynomial* (1.2)*, we call* f *a* quasi-ordinary *polynomial with respect to* z *if the discriminant* Δ *is of the form*

$$\Delta(x_1, x_2) = x_1^{p_1} x_2^{p_2} V(x_1, x_2), \quad p_1, p_2 \in \mathbb{N}, \tag{1.3}$$

where V *is an analytic function such that* $V(0,0) \neq 0$*.*

Now, the existence of a parametric equation that satisfies the equation $f(z, \mathbf{x}) = 0$ for quasi-ordinary polynomials, i.e., the existence of Puiseux series solutions in the multi-parameter case is given by the following (see, for instance, [93, 52]):

Theorem 1.3.3 (Abhyankar-Jung Theorem)**.** *Consider* $f \in \mathbb{C}\{x_1, x_2\}[z]$ *be a quasi-ordinary polynomial in* z *with analytic coefficients in* $\mathbb{C}\{x_1, x_2\}$*. Then, there exist a natural number* r *such that the roots of* f *belong to the field of convergent fractional power series* $\mathbb{C}\{x_1^{1/r}, x_2^{1/r}\}$*.*

The condition for a quasi-ordinary polynomial is best explained with an example, through the asymptotic behavior of a quasi-polynomial root as follows.

Example 1.3.3. *The roots of the following quasi-polynomial are given by convergent power series in* $\mathbb{C}\{x_1^{1/2}, x_2^{1/2}\}$*:*

$$P(z, x_1, x_2) = z^4 - (2x_1 + 2x_1^2 x_2) z^2 + (x_1^2 - 2x_1^3 x_2 + x_1^4 x_2^2).$$

Due to its form, the discriminant of the polynomial can be easily calculated, so that

$$\Delta(x_1, x_2) = 4096 x_1^6 x_2^2 \left(x_1^4 x_2^2 - 2x_1^3 x_2 + x_1^2 \right).$$

In Chapter 3 a method for the computation of the Puiseux series solutions will be proposed. The four roots $z_i = \phi_i(x_1, x_2)$ *of the quasi-ordinary polynomial are as follows:*

$$z_1(x_1, x_2) = x_1^{1/2} + x_1 x_2^{1/2}, \qquad z_2(x_1, x_2) = x_1^{1/2} - x_1 x_2^{1/2} \tag{1.4}$$

$$z_3(x_1, x_2) = -x_1^{1/2} + x_1 x_2^{1/2}, \qquad z_4(x_1, x_2) = -x_1^{1/2} - x_1 x_2^{1/2}. \tag{1.5}$$

Remark 1.3.2. *In the following sections, various theorems and methods about the behavior of analytic functions in the neighborhood of a simple or a multiple point. While the behavior of singular roots represents a challenge on stability analysis, we present a discussion in Section 4.8 for the study of certain singular points.*

1.4 Newton Diagram Method

In order to simplify the presentation, we will assume in the sequel that $z_0 = 0$ is a zero of multiplicity $m > 1$ of $f(z, \mathbf{x})$ at $\mathbf{x}_0 = 0$, making appropriate shifts $z \mapsto z - z_0$, $x \mapsto x - \mathbf{x}_0$ if necessary.

For a better understanding of asymptotic behavior, some subtleties are presented that must be taken into account in the study of multiple roots. For this purpose, we define an efficient method to compute power series solutions of the equation $f(z, x) = 0$. The following theorem allows us to deal with this situation and allows us to find multiple roots of $f(z, x) = 0$ for z in terms of x.

Theorem 1.4.1 (Puiseux Theorem, [90]). *The equation $f(z, x) = 0$, with f given in formal power series such that $f(0, 0) = 0$, possess at least one solution in power series of the form:*

$$x = t^q, \quad z = \sum_{i=1}^{\infty} c_i t^i, \quad q \in \mathbb{N}. \tag{1.6}$$

The above theorem is regularly called in the literature as Newton-Puiseux algorithm due to historical reasons. Isaac Newton described an iterative method to computed terms of the solution of the equation $f(z, x) = 0$ in a later [71] but the existence of this power series solution was proven two centuries later by Puiseux in [77].
Equation (1.6) gives the parameterized solution which can also be uniformly represented as convergent fractional power series. This is done by making the q-root of x obtaining $t = x^{1/q}$ in such a way that

$$z(x) = \sum_{i=1}^{\infty} c_i x^{i/q}.$$

Now, the local behavior of a solution $z(\mathbf{x})$ can be obtained by means of the *Newton-Diagram Method*. This method is named after Isaac Newton around 1670 [71] but until the 19th century was used by Puiseux to prove the Theorem 1.4.1 (see [77]), and for this reason, this method is also called the Newton-Puiseux algorithm.

The basic idea is to construct a power series solution term by term in a simple manner. Before giving a description of this method, we consider the following definition.

Definition 1.4.1. *Let* $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \mathbf{x}^\alpha$ *be a formal power series. The support of* f *is the set of exponents:*

$$sup(P) = \{\alpha \in \mathbb{Z} \mid c_\alpha \neq 0\}.$$

For simplicity, let f be a *pseudo-polynomial* in z with support $sup(f) \subset \mathbb{Q}^2$, i.e.,

$$f(z, x) = a_n(x)z^n + a_{n-1}(x)z^{n-1} + \cdots + a_0(x), \tag{1.7}$$

where the corresponding coefficients are given by

$$a_k(x) = x^{\,\rho_k} \sum_{r=0}^{\infty} a_{rk} x^{r/q}, \quad a_{rk} \in \mathbb{C}, \rho_k \in \mathbb{Q}_+ \tag{1.8}$$

with $a_{rk} \in \mathbb{C}$, x and z are complex variables, ρ_k are non-negative rational numbers, q is an arbitrary natural number, and the leading term is such that $a_n(x) \not\equiv 0$.

Beforehand, we know that the solutions must have a particular form, thus, the following series of powers is assumed as a solution of $f(z, x) = 0$ (1.7):

$$z(x) = c_1 x^{\epsilon_1} + c_2 x^{\epsilon_2} + c_3 x^{\epsilon_3} + \cdots, \tag{1.9}$$

since we are considering roots around the origin, the term c_0 is discarded, and the exponents satisfy $\epsilon_1 < \epsilon_2 < \epsilon_3 < \cdots$ and $c_1 \neq 0$. To determine the values of ϵ_i and c_1, the *Newton's diagram* (see Figure 1.1) is considered.

Definition 1.4.2 (Newton's Diagram and Polygon)**.** *Given a pseudo-polynomial of the form* (1.7)*, the Newton Diagram is the convex hull of its support* $sup(f)$*, and the Newton Polygon will be given by the lower boundary of the convex hull.*

Remark 1.4.1. *The Newton polygon is made of a finite number of straight-line segments, which do not lie on the axis.*

The method can be structured by the following steps.

step 1 Define the Newton Diagram Π as the set of points $\pi_k = (k, \rho_k)$, denoted by Π:

$$\Pi = \{\pi_k \,:\, a_k(\cdot) \neq 0\}\,;$$

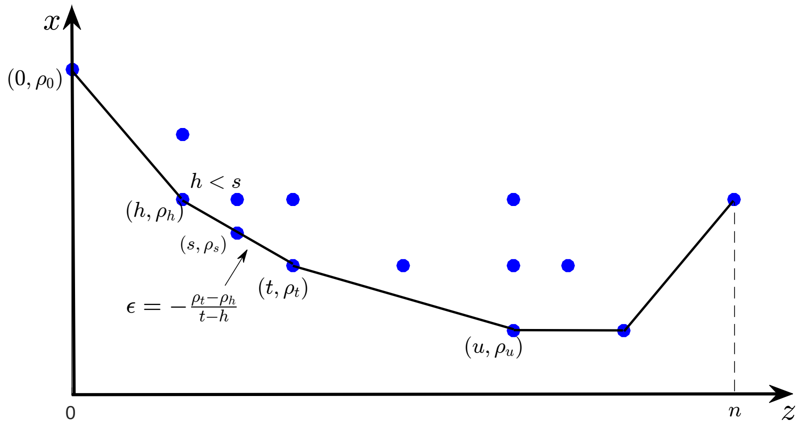


Figure 1.1: The points $\pi_k = (k, \rho)$ depicted by blue stars determine the Newton Diagram Π. The union of the black lines segments, with slope $(\rho_t - \rho_h)/(t - h)$, form The Newton Polygon for the pseudo-polynomial $f(z, x)$ (1.7).

step 2 Plot k versus ρ_k for k for $k = 0, 1, \ldots, n$, if $a_k(\cdot) \equiv 0$, the corresponding point is disregarded;

step 3 Define the Newton Polygon by the lower boundary of the convex hull of the set Π;

step 4 Take a segment of the Newton polygon, then the first exponent ϵ_1 will be its negative slope;

step 5 The first coefficient c_1 will be the non-zero root of the polynomial form by lower order terms on x:

$$\sum_{l=i}^{j} a_{0,l}\xi^l = 0$$

step 6 For higher order terms, ϵ_j and c_j, repeat the process for $f_{j+1}(z_{j+1}, x)$ with $z_j = c_j + z_j$.

For a given pseudo-polynomial f, Fig. 1.1 simply illustrates Definition 1.4.2.

Example 1.4.1. *In this example, we compute the expansion of the roots of the following polynomial*

$$p(z, x) = z^3 + z^2 + xz + 2x^2 = 0, \tag{1.10}$$

up to the first approximation. By the Puiseux Theorem 1.4.1, this polynomial has solutions of the form

$$z(x) = c_1 x^{\epsilon_1} + c_2 x^{\epsilon_2+\epsilon_2} + \cdots .$$

Following the steps mentioned above. The Newton Diagram of p is given the set:

$$\Pi = \{(2,0),(1,1),(0,2),(0,3)\},$$

which is plotted in Fig. 1.2. Two line segments compose the Newton Polygon, the first with slope $-\epsilon_0 = 1$ *and the second with slope* $-\epsilon_1 = 0$*. Now, lets consider*

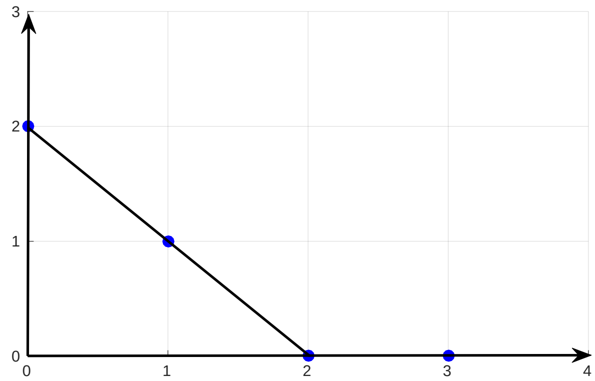


Figure 1.2: The Newton Diagram and Polygon for the polynomial $p(z,x)$ (1.10).

the segment with slope $\epsilon_1 = 0$*. In order to obtain the coefficient associated with the exponent* ϵ_1*, the points* π_k *on this segment (lower-order terms) determine the polynomial equation*

$$\xi^2 + \xi^3 = 0,$$

with 0 *a root of multiplicity* 2 *and* -1 *a simple root. Thus the coefficient* c_1 *associated with this segment is*

$$c_1 = -1.$$

The second segment, with slope $\epsilon_2 = 1$*, has the following associated polynomial*

$$2x^2 + x\xi + \xi^2 = 0,$$

with roots. Thus, set of solutions is given by

$$z_1(x) = -1 + \mathcal{O}(x),$$
$$z_2(x) = -0.5\left(1 + \boldsymbol{i}\sqrt{7}\right)x + \mathcal{O}(x^2),$$
$$z_3(x) = -0.5\left(1 - \boldsymbol{i}\sqrt{7}\right)x + \mathcal{O}(x^2).$$

1.5 Generalized Puiseux Series and Cones

In the preceding pages, the Weierstrass preparation Theorem 1.3.2 and Puiseux Theorem 1.4.1 were presented as the main tool for the local representation and determination of multiple roots of the algebraic equation $f(z, \mathbf{x}) = 0$. In order to give the asymptotic behavior of multiple roots, *multi-variable fractional Puiseux series* are used for this purpose.
In the case of multi-parameter algebraic equations, we deal with singularities of greater dimension, accordingly, we must use a ring of multi-variable fractional power series. In [63] the author defines a fractional power series ring that contains the solutions of algebraic hyper-surfaces. This can be achieved through formal power series defined in a geometric way, by taking infinite power series:

$$\sum_{i=1}^{\infty} c_{\mathbf{a_i}} \mathbf{x}^{\mathbf{a}_i/d}, \qquad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n},$$

where the exponents **a** are taken from fixed convex cones with a structure related to its Newton polytopes [15].

Definition 1.5.1. *A convex polyhedral cone is a set of the form*

$$\mathscr{C} = \left\{ \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \in \mathbb{R}_+ \right\},$$

where $M = \{v_1, \ldots, v_m\}$ is a finite set of vectors in $\mathbb{R}^n$.

A convex polyhedral cone in $\mathbb{R}^3$ is shown in figure 1.3 We will use fractional iterated power series of several variables as *Generalized Puiseux Power Series* (see [86, 70]), denoted by $K_{\mathbf{x},d}$. This series can be constructed by induction, taking as a bases the univariate case $K_{x_1,d}$ and then, proceed with the field of power series in $x_1^{1/d}$ with power series coefficients in $x_2^{1/d} \cdots x_n^{1/d}$ such that

$$K_{\mathbf{x},d} := \mathbb{C}[[x_n^{1/d}]]\left[\left[x_{n-1}^{1/d}\right]\right] \cdots \left[\left[x_1^{1/d}\right]\right].$$

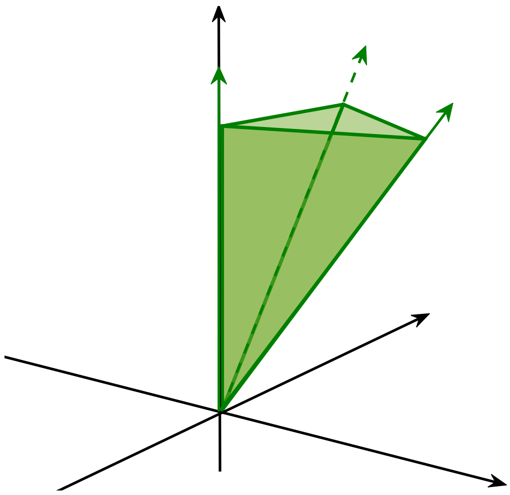


Figure 1.3: A convex polyhedral cone in $\mathbb{R}^3$.

1.6 Chapter Summary

In this chapter, we have introduced the mathematical tools necessary for the asymptotic analysis of roots. A distinction is made between two types of algebraic function points, which allows us to distinguish between multiple roots (analyzed in Chapter 2) and roots with singular behavior (discussed in Section 4.8). Through the presented local approach, the Weierstrass polynomial $W(z, \mathbf{x})$, the asymptotic analysis of the multiple roots under multi-parameter variations will be addressed in Chapter 3.

Chapter 2

Weierstrass Approach to Asymptotic Behavior

The essential problem addressed in this chapter concerns the development of an analytical and efficient method to find explicitly the solutions of the equation $f(s,\tau)=0$ around some critical values. In particular, a method for the asymptotic analysis of multiple roots for quasi-polynomials for small deviations of the time-delay is given.
The chapter is structured as follows. First, a motivational example is introduced to motivate further development. Then, the asymptotic analysis of simple roots is presented as a basis for the analysis of multiple roots. In Section 2.3, the Weierstrass polynomial is used as a local form of the characteristic function, by means of an analytic and a numerical approach. Next, a method for the asymptotic analysis is provided, by given a method for the computation of the Puiseux series solutions together with a classification of the splitting of the branching solutions around critical points in Section 2.4. Several examples of time-delay systemss are given illustrating how to apply the results are given at the end of the chapter.

2.1 Polynomial Root Perturbation

In this section, we introduce the main idea of the study of a system under a parametric perturbation. Let's start with an example. For the sake of simplicity, consider an algebraic equation $P(z,x)=0$.

Example 2.1.1. *Determine the dependency of the roots of $P(z, x)$ which can be written as*

$$z^3 + (2 - x)z^2 - (1 + 2x)z - 2 = 0$$

on the parameter x. First, considering the case free of perturbation, $x = 0$, it is easy to find the roots, $z_1 = 1$, $z_2 = -1$ and $z_3 = -2$.
Next, for small positive x, an invariant root z_3, meaning that is independent of variations of the parameter x (later in this chapter this case will be described). For the remaining two roots:

$$z_1 = -\frac{x}{2} + \sqrt{1 + \frac{x^2}{4}},$$
$$z_2 = -\frac{x}{2} - \sqrt{1 + \frac{x^2}{4}},$$

as a direct consequence of the quadratic formula. As a first approximation, the Taylor series expansion of these two exact solutions is given by the first terms as follows

$$z_1 = 1 + \frac{x}{2} + \frac{x^2}{8} + \mathcal{O}(x^3),$$
$$z_2 = -1 + \frac{x}{2} - \frac{x^2}{8} + \mathcal{O}(x^3).$$

From the example above the need and motivation for the analysis of the effect of the parameter on the roots with multiplicity $m > 1$, and to give conditions in the appearance of fixed roots. In this vein, the main purpose of the thesis is to analyze the effect of the time-delay on the roots.
By means of the Weierstrass Preparation Theorem 1.3.2 the complexity of the analysis of the system's characteristic function can be reduced to analyze some algebraic properties of a given polynomial (known as Weierstrass polynomial) with a degree equal to the multiplicity of the critical zero. Furthermore, it is well known that such Weierstrass polynomial (for further details, see [37]) preserves the full information concerning the stability behavior. Such an approach has been adopted by [20], where by means of the *calculus of residues* (see, for instance, [81]), the authors have proposed an analytical method to construct such a polynomial. It is worth mentioning that the corresponding contribution does not focus on deriving

an analytical characterization of the solutions or any characterization of the local behavior of the characteristic roots with respect to the delay parameter variation. Next, another important contribution in this direction is proposed by [43], where the authors found an explicit formula (see, for instance, Theorem 4) to characterize the local behavior for the first-order terms in the case of a double characteristic root. The asymptotic behavior of multiple critical roots has also been considered by [50, 51], where, instead of computing the Weierstrass polynomial, the authors derived the Puiseux series expansion of first-order by a direct application of the Newton diagram procedure. Their results allow computations in an efficient way. Finally, more recently, in [49], the asymptotic behavior analysis of critical solutions with respect to infinitely many critical delays (including the case of one delay) has been considered where, in particular, the authors solve the general invariance property treated under some appropriate constraints in [43].

Before starting with the main result, we present a result for simple root which allows the asymptotic analysis for simple roots in the next section.

2.2 Asymptotic Behavior for Simple Roots

Before continuing with the analysis of the multiple roots we give a result to deal with simple roots. Following the τ-decomposition method to stability analysis we need to look for roots on the imaginary axes of the quasi-polynomial such that the number of roots on the right-hand plane changes. Thus the crossing direction of the critical roots should be determined. When the first terms of the series expansion do not give enough information about the crossing pattern as shown in Section , we need to compute higher-order terms. The following proposition allows solving this problem by computing higher-order terms of the Taylor expansion solutions around simple roots.

Proposition 2.2.1. *Let the quasi-polynomial $f(s,\tau)$ (5) be expressed in Taylor form:*

$$f(s,\tau) = \sum_{i+j=1}^{\infty} a_{ij}\tau^i s^j,$$

with a simple critical root $s = \boldsymbol{i}\omega$ at $\tau = \tau^*$ i.e.,

$$f(s^*, \tau^*) = 0, \quad \frac{\partial f}{\partial s} \neq 0 \Big|_{(s^*,\tau^*)} .$$

Then around the point (s^*, τ^*) there exists a convergent power series solution

$$\phi(\tau) = \sum_{i=1}^{\infty} c_i \tau^i,$$

where the coefficients $c_i \in \mathbb{C}$ are given by

$$c_i = -\frac{a_{i,0}}{a_{0,1}} - \sum_{j=1}^{i-1} \frac{a_{j,1}}{a_{0,1}} c_{i-j} - \left[\frac{a_{0,i}}{a_{0,1}} c_1^i + \sum_{j=0}^{i-2} \sum_{k=2}^{i-2} \sum_{l=1}^{i-2} \frac{a_{\cdot jk}}{a_{0,1}} \Pi_{m=1}^{i-1} \binom{l}{m} c_m \right]$$

Proof. We can assume without loss of generality that $a_{0,1} = 1$, then the Taylor series of f can be written as

$$f(s, \tau) = s + \sum_{i=1}^{\infty} \left[a_{i,0} + a_{i,1} s\right] \tau^i + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} a_{ij} \tau^i s^j.$$

If we make $b_{ij} = -a_{ij}$, the implicit function $f(s, \tau) = 0$ can be rewrite as

$$s = \sum_{i=1}^{\infty} \left[b_{i,0} + b_{i,1} s\right] \tau^i + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} b_{ij} \tau^i s^j.$$

By the implicit function theorem, there exists a convergent power series $\phi(\tau)$ such that

$$\sum_{i=1}^{\infty} c_i \tau^i = \sum_{i=1}^{\infty} b_{i,0} \tau^i + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{i,1} c_j \tau^{i+j} + \sum_{i=0}^{\infty} \sum_{j=2}^{\infty} b_{ij} \tau^i \left(\sum_{k=1}^{\infty} c_k \tau^k \right)^j . \tag{2.1}$$

Since ϕ is an absolutely convergent power series in an appropriate neighborhood of (s^*, τ^*), we are able to rearrange in order to equate like powers of τ without changing the sum. □

As mentioned early simple roots can be represented by Taylor power series of arbitrary order. The order of the series can be determined as follows.s

Corollary 2.2.1. *Let $s = 0$ be a simple root at $\tau = 0$ of the quasi-polynomial $f(s, \tau)$. If*

$$f(0,0) = \left.\frac{\partial f}{\partial \tau}\right|_{(0,0)} = \cdots \left.\frac{\partial^{n_0-1} f}{\partial \tau^{n_0}}\right|_{(0,0)} = 0 \ \left.\frac{\partial^{n_0} f}{\partial \tau^{n_0}}\right|_{(0,0)} \neq 0,$$

then the order of Taylor series solutions is n_0, with the form

$$s(\tau) = c_{n_0}\tau^{n_0} + \cdots .$$

It is worth mentioning that in the case of simple critical roots are continuous functions of the delay, making it possible to use the implicit function. In what follows, we deal with multiple roots, where differentiability is lost.

2.3 Algebraic Reduction

The asymptotic behavior of the critical zeros of the quasi-polynomial $f(s, \tau)$ will be performed by means of the Newton diagram procedure. Since this method is defined for bivariate polynomials in Section 1.4 the quasi-polynomial is replaced by the Weierstrass polynomial in Section 1.3.2. To this end, since any critical solution (s^*, τ^*) can always be translated to the origin by appropriate shifts $s \mapsto s - s^*$, $\tau \mapsto \tau - \tau^*$, hereinafter we will assume that $(s^*, \tau^*) = (0, 0)$. The following numerical approximation to the computation of the Weierstrass polynomial is based on the work done in [61].

2.3.1 The Weierstrass Polynomial. Numerical Approximation

Since the quasi-polynomial $f(s, \tau)$ is an analytic function; it is not difficult to see that f can be expressed as

$$f\left(s,\tau\right) = \sum_{i=0}^{\infty} f_i\left(\tau\right) s^i, \tag{2.2}$$

with f_i given as

$$f_i\left(\tau\right) := \frac{1}{i!}\sum_{j=i}^{\infty} \frac{1}{(j-i)!} \left.\frac{\partial^j f}{\partial s^i \partial \tau^{j-i}}\right|_{(0,0)} \tau^{j-i}.$$

As suggested by [37], let us express f as follows:

$$f(s,\tau) = l(s,\tau) + s^m h(s,\tau), \tag{2.3}$$

where $l(s,\tau)$ and $h(s,\tau)$ retain the lower and higher order terms of f, respectively, i.e.,

$$l(s,\tau) := \sum_{j=0}^{m-1} f_j(\tau)\, s^j, \quad h(s,\tau) := \sum_{j=m}^{\infty} f_j(\tau)\, s^{j-m}.$$

Clearly, both l and h are holomorphic functions. Furthermore, h is non-vanishing and analytic in a neighborhood of $0 \in \mathbb{C} \times \mathbb{R}$. From the Weierstrass Preparation Theorem we have:

$$W(s,\tau)\, b(s,\tau) = l(s,\tau) + s^m h(s,\tau), \tag{2.4}$$

where, according to this result, $W(s,\tau) = s^m + w_{m-1}(\tau)s^{m-1} + \cdots + w_0(\tau)$, b is holomorphic and $b(0,0) \neq 0$, implying that b^{-1} is analytic in a neighborhood of the origin. Thus, (2.4) can be rewritten as:

$$W(s,\tau) = \left(s^m + \frac{l}{h}\right) hb^{-1}, \tag{2.5}$$

Since W is monic, then it can be written as $W(s,\tau) := s^m - \widehat{W}(s,\tau)$, and defining $\varphi := hb^{-1}$, (2.5) can be expressed as:

$$\begin{aligned} s^m - \widehat{W} &= \left(s^m + \frac{l}{h}\right)\varphi, \\ \Rightarrow s^m - \frac{l}{h}\varphi &= s^m\varphi + \widehat{W}. \end{aligned} \tag{2.6}$$

The equation (2.6) can be solved by successive approximations, to this end it can be written as:

$$s^m - \frac{l}{h}\varphi_{k-1} = s^m\varphi_k + \widehat{W}_k, \tag{2.7}$$

where $\widehat{W}_k$ is a polynomial in s of degree $< m$. For $k=1$ set $\varphi_0 := 0$ getting

$$s^m = s^m\varphi_1 + \widehat{W}_1,$$

that clearly imposes the following initial conditions $\varphi_1 = 1$ and $\widehat{W}_1 = 0$, or equivalently $\varphi_1 = 1$ and $\widehat{W}_0 = 0$. For $k = 1, 2, \ldots$, $\widehat{W}_k$ can be obtained as the remainder after left-hand side (2.7) is divided by s^m.

Hence, the following result is a slight modification of the Weierstrass Preparation Theorem presented in [4, 45].

Proposition 2.3.1. *Let $f(s,\tau)$ be a quasi-polynomial written as* (2.3)*, such that $s = 0$ is a zero of multiplicity m, with $m > 1$. Assume that $f(0,\tau)/s^m$ is holomorphic in a neighborhood of $0 \in \mathbb{C}$ and for $k = 1, 2, \ldots$, let $\widehat{W}_k(s,\tau)$ be obtained by the procedure* (2.7)*. Then, Weierstrass polynomial is given by*

$$W(s,\tau) = s^m - \widehat{W}(s,\tau),$$

where

$$\widehat{W}(s,\tau) = \lim_{k\to\infty} \widehat{W}_k(s,\tau).$$

Remark 2.3.1. *Even though the above result seems to be a powerful tool to derive the Weierstrass polynomial, it has the inconvenience that it requires an infinite number of iterations to obtain the exact Weierstrass polynomial. However, since the focus in analyzing the stability behavior for the solutions around the critical pair, then, according to the Newton procedure, it is only necessary to know the leading terms of each $a_j(\tau)$.*

In the light of Remark 2.3.1, let us adopt the following notation for $\widehat{W}$:

$$\widehat{W}(s,\tau) = w_{m-1}(\tau)\, s^{m-1} + w_{m-2}(\tau)\, s^{m-2} + \cdots + w_0(\tau),$$

with

$$w_i(\tau) := \tau^{\rho_i} \sum_{j=0}^{\infty} w_{i,j}\tau^j. \tag{2.8}$$

While for its $k-$th approximation, $\widehat{W}_k$ will be expressed as:

$$\widehat{W}_k(s,\tau) = w_{m-1}^{(k)}(\tau)\, s^{m-1} + w_{m-2}^{(k)}(\tau)\, s^{m-2} + \cdots + w_0^{(k)}(\tau),$$

where

$$w_i^{(k)}(\tau) := \sum_{j=1}^{\infty} w_{i,j}^{(k)}\tau^j. \tag{2.9}$$

Bearing in mind the previous notations, the following remark holds:

Remark 2.3.2. *Let* $\operatorname{ord}_\tau(w_i) = \rho_i$ *for* $i = 0, \dots, m-1$. *Note that there may be cases in which* $\rho_0, \rho_1 = \cdots = n_{\kappa-1} = \infty$ *and* $\rho_\kappa < \infty$ *implying* $w_i(\tau) \equiv 0$ *for* $i < \kappa$ *with* $\kappa \in \mathbb{N}$. *Hence, the polynomial* $\widehat{W}$ *is reduced to*

$$\widehat{W}(s,\tau) = w_{m-1}(\tau)s^{m-1} + w_{m-2}(\tau)s^{m-2} + \cdots + w_\kappa(\tau).$$

Then, ρ_κ *satisfies the following relations:*

$$\frac{\partial^\kappa f}{\partial s^\kappa} = \frac{\partial^{\kappa+1} f}{\partial \tau \partial s^\kappa} = \cdots = \frac{\partial^{\kappa+\rho_\kappa-1} f}{\partial \tau^{\rho_\kappa-1} \partial s^\kappa} = 0, \quad \frac{\partial^{\kappa+\rho_\kappa} f}{\partial \tau^{\rho_\kappa} \partial s^\kappa} \neq 0, \tag{2.10}$$

where the partial derivatives are taken at $(0,0)$. *In fact, since* f *is analytic at* $(0,0)$, *according to the Weierstrass Preparation Theorem, in a neighborhood of* $(0,0)$, *we have that* $f(s,\tau) = W(s,\tau)b(s,\tau)$ *where* $b(0,0) \neq 0$. *Then, by* (1.2) *we have that* $f(0,\tau) = a_\kappa(\tau)b(0,\tau)$ *with* $b(0,\tau) = b_0 + \sum_{i=1}^\infty b_i\tau^i$. *Thus, for* $i < \rho_\kappa$

$$\left.\frac{\partial^{\kappa+i} f(0,\tau)}{\partial s^\kappa \partial \tau^i}\right|_{\tau=0} = \frac{\partial^{\kappa+i}}{\partial s^\kappa \partial \tau^i}\left.\left[a_\kappa(\tau)\left(b_0 + \sum_{i=1}^\infty b_i\tau^i\right)\right]\right|_{\tau=0},$$

the right-hand side of the above expression must be equal to zero.

From the above discussion, the following result is obtained.

Proposition 2.3.2. *Let* $s^* = 0$ *be an* $m-$*multiple zero of* $f(s,\tau)$ *at* $\tau^* = 0$ *and assume* $\kappa = 0$ (2.10). *Then, the full stability information for the* $m-$*zeros* $s_\ell(\tau)$, *with* $\ell = 1, \dots, m$ *is completely determined by* $W(s,\tau) \approx s^m - \widehat{W}_k(s,\tau)$ *for* $k = \rho_0$.

Proof. Following the recursion (2.5), first, it is necessary to compute l/h

$$\frac{l(s,\tau)}{h(s,\tau)} = \sum_{i=0}^\infty q_i(\tau)s^i, \tag{2.11}$$

where simple computations reveal that the coefficients q_i can be recursively expressed as:

$$\begin{aligned} f_m(\tau)\sum_{i=0}^\infty q_i(\tau)s^i = &\sum_{i=0}^{m-1}\left(f_i - \sum_{k=0}^{i-1} q_k f_{m+i-k}\right)s^i - \left(\sum_{k=0}^{m-1} q_k f_{m-1-k}\right)s^m \\ &- \sum_{i=m+1}^\infty \left(\sum_{k=0}^{i-2} q_k f_{m+i-k} + q_{i-1}f_{m+1}\right)s^i, \end{aligned} \tag{2.12}$$

where $q_0 = f_0/f_m$. Now, since each q_i depends solely on the f_j's, it will be useful to determine the order of each f_j. From (2.2), it is evident that $\text{ord}\,(f_j) \geq 1$ $\forall j \in \mathbb{N} \cup \{0\}$. Since $\kappa = 0$, from Remark 2.3.2 we know that $\text{ord}_\tau\,(f_0) = \rho_0$.
Then, the result will be proven if we are able to show that after ρ_0-iterations the coefficients of the approximation $W\,(s,\tau) \approx s^m - \widehat{W}_k\,(s,\tau)$ that fall in the convex hull of the Newton diagram remain unchanged for all $k \geq \rho_0$. Bearing in mind the above observations, let us introduce the following:

$$\text{ord}_\tau\,(q_0) \equiv \rho_0; \qquad \text{ord}_\tau\,(q_i) := \rho_{q_i} \geq 1,\ \forall i \neq m; \qquad \text{ord}_\tau\,(q_m) := \rho_{q_m} \geq 2.$$

Next, from (2.7), we consider $\widehat{W}_k$ for $k = 1, 2, 3$. Thus, we obtain:

$$\begin{aligned}
\widehat{W}_1\,(s,\tau) &= -\sum_{i=0}^{m-1} q_i s^i, \quad \widehat{W}_2\,(s,\tau) = \widehat{W}_1 + \sum_{i=0}^{m-1}\left(\sum_{j=0}^{i} q_{i-j}q_{m+j}\right)s^i, \\
\widehat{W}_3\,(s,\tau) &= \widehat{W}_2 - \sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}\sum_{k=0}^{m+j} q_{i-j}q_{m+j-k}q_{m+k}\right)s^i.
\end{aligned}$$

From the above expressions, we have that $w_i^{(k)}$ is given as:

$$w_i^{(1)} = -q_i; \quad w_i^{(2)} = w_i^{(1)} + \tilde{w}_i^{(2)}; \quad w_i^{(3)} = w_i^{(2)} - \tilde{w}_i^{(3)},$$

where

$$\tilde{w}_i^{(2)} := \sum_{j=0}^{i} q_{i-j}\,q_{m+j}; \qquad \tilde{w}_i^{(3)} := \sum_{j=0}^{i}\sum_{k=0}^{m+j} q_{i-j}\,q_{m+j-k}\,q_{m+k}.$$

From the previous expressions it is clear that $\text{ord}_\tau\left(w_i^{(1)}\right) \equiv \rho_{q_i}$, whereas $\text{ord}_\tau\left(w_i^{(2)}\right)$ and $\text{ord}_\tau\left(w_i^{(3)}\right)$ depends on $\text{ord}_\tau\left(\tilde{w}_i^{(2)}\right)$ and $\text{ord}_\tau\left(\tilde{w}_i^{(3)}\right)$, respectively. Hence, the order of these expressions are given as follows:

$$\begin{aligned}
\text{ord}_\tau\!\left(\tilde{w}_i^{(2)}\right) &= \min\left\{\rho_{q_i} + \rho_{q_m},\ \rho_{q_{i-1}} + \rho_{q_{m+1}}, \ldots, \rho_{q_1} + \rho_{q_{m+i-1}},\ \rho_{q_0} + \rho_{q_{m+i}}\right\}, \\
\text{ord}_\tau\!\left(\tilde{w}_i^{(3)}\right) &= \min\left\{\rho_{q_i} + 2\rho_{q_m},\ \rho_{q_i} + 2\rho_{q_{m+1}}, \ldots, \rho_{q_0} + \rho_{q_1} + \rho_{q_{2m+i-1}},\ 2\rho_{q_0} + \rho_{q_{2m+i}}\right\}.
\end{aligned}$$

Thus, the following inequalities hold:

$$\text{ord}_\tau\!\left(\tilde{w}_i^{(2)}\right) \leq \text{ord}_\tau\!\left(w_i^{(1)}\right) \quad \text{and} \quad \text{ord}_\tau\!\left(\tilde{w}_i^{(2)}\right) < \text{ord}_\tau\!\left(\tilde{w}_i^{(3)}\right),$$

implying that

$$\mathrm{ord}_\tau\Big(w_i^{(2)}\Big) = \min\Big\{\mathrm{ord}_\tau\Big(w_i^{(1)}\Big), \mathrm{ord}_\tau\Big(\tilde{w}_i^{(2)}\Big)\Big\} \quad \text{and} \quad \mathrm{ord}_\tau\Big(w_i^{(3)}\Big) = \mathrm{ord}_\tau\Big(w_i^{(2)}\Big).$$

For the next steps ($k > 3$), from the iterative construction (2.7) it is clear that $w_i^{(k)}$ is given as

$$w_i^{(k)} = w_i^{(k-1)} + (-1)^k \tilde{w}_i^{(k)}.$$

Moreover, since q_j (for some $j \in \mathbb{N} \cup \{0\}$) is always a factor of $\tilde{w}_i^{(k)}$ we will have the following relations:

$$\begin{aligned} \mathrm{ord}_\tau\Big(\tilde{w}_i^{(k)}\Big) &< \mathrm{ord}_\tau\Big(\tilde{w}_i^{(k+1)}\Big), \\ \Rightarrow \mathrm{ord}_\tau\Big(w_i^{(k)}\Big) &= \mathrm{ord}_\tau\Big(w_i^{(k+1)}\Big). \end{aligned}$$

Then, from the above discussion, we can conclude that all coefficients $w_{i,j}$ belonging to the convex hull of $W(s,\tau)$ will stop updating at most at the ρ_0-th step, leading thus to the expected result. □

Remark 2.3.3. *Generally, we have $\kappa < m$ and ρ_κ. Now, following the above proof, it can be seen that ρ_κ represents an upper bound that ensures the computation of the exact leading coefficient of $W(s,\tau)$ that belongs to the convex hull of the Newton diagram. However, it is worth mentioning that such coefficients can be obtained in a fewer number of steps.*

2.3.2 The Weierstrass Polynomial. Analytical Approach

An alternative procedure for the computation of the Weierstrass polynomial is using recursive relation for the partial derivatives of the quasi-polynomial $f(s,\tau)$. First remember that, for an $m-$multiple root $s = 0$ of f at $\tau = 0$, according to the Weierstrass Preparation Theorem we will have that:

$$f(s,\tau) = \big(s^m + w_{m-1}(\tau)s^{m-1} + \cdots + w_0(\tau)\big)\, b(s,\tau). \tag{2.13}$$

Now, with the aim of avoiding unnecessary computations, the following notations will be useful. For $i \in \{0, 1, \ldots, m-1\}$, let $n_i \in \mathbb{N}$ denote the first nonzero partial derivatives in (s,τ) of f at $(0,0)$, such that the following relations hold:

$$f(0,0) = \left.\frac{\partial^i f}{\partial s^i}\right|_{(0,0)} = \cdots = \left.\frac{\partial^{i+n_i-1} f}{\partial s^i \partial \tau^{n_i-1}}\right|_{(0,0)} = 0, \qquad \left.\frac{\partial^{i+n_i} f}{\partial s^i \partial \tau^{n_i}}\right|_{(0,0)} \neq 0. \tag{2.14}$$

Even though the following result is a straightforward application of the Taylor series, it will be extremely useful in the sequel.

Lemma 2.3.1. *Let $s = 0$ be an m-multiple root at $\tau = 0$ of the quasi-polynomial $f(s, \tau)$, and let $f = Wb$ be defined as in* (1.2) *from Chapter 1. Then, the following statements hold:*

(i) the first nonzero partial derivatives with respect to τ are given by:

$$\left.\frac{\partial^{n_0} f}{\partial \tau^{n_0}}\right|_{(0,0)} = \left.\frac{\partial^{n_0} w_0}{\partial \tau^{n_0}}\right|_{(0,0)} b(0,0);$$

(ii) the first nonzero partial derivatives with respect to s and τ for $i = 1, \ldots, m-1$ are given by:

$$\left.\frac{\partial^{i+n_i} f}{\partial \tau^{n_i} \partial s^i}\right|_{(0,0)} = \sum_{j=0}^{i} \left.\left[j! \binom{i}{j} \sum_{k=0}^{n_i} \binom{n_i}{k} \frac{\partial^{n_i - k} w_j}{\partial \tau^{n_i - k}} \frac{\partial^{i-j+k} b}{\partial \tau^k \partial s^{i-j}} \right]\right|_{(0,0)};$$

and,

(iii) the $m-$derivatives with respect to s at the multiple critical point satisfy:

$$\left.\frac{\partial^m f}{\partial s^m}\right|_{(0,0)} = m! b(0,0).$$

Proof. According to Theorem 1.3.2, we have that f admits the representation given in (2.13). Based on the previous observations, let us consider in the remaining n_i as given in (2.14).

(i) It is not difficult to see that the first $r-$partial derivatives with respect to τ are given by:

$$\frac{\partial^r f}{\partial \tau^r} = \sum_{j=0}^{r} \binom{r}{j} \frac{\partial^{r-j} w_0}{\partial \tau^{r-j}} \frac{\partial^j b}{\partial \tau^j} + \frac{\partial^r}{\partial \tau^r} \sum_{j=1}^{m-1} w_j s^j b.$$

Now, observe that

$$\left.\frac{\partial^r}{\partial \tau^r} \sum_{j=1}^{m-1} w_j s^j b \right|_{(0,0)} = 0, \ \forall r \geq 0, \quad \text{and} \quad \left.\sum_{j=0}^{r} \binom{r}{j} \frac{\partial^{r-j} w_0}{\partial \tau^{r-j}} \frac{\partial^j b}{\partial \tau^j}\right|_{(0,0)} = 0, \ \forall r < n_0.$$

Thus, it is clear that for $r = n_0$ and $(s, \tau) = (0, 0)$, we get the desired result.

(ii) Following similar steps to those presented in (i), one gets:

$$\frac{\partial^{r+n} f}{\partial\tau^n \partial s^r} = \sum_{j=0}^{r}\left[j!\binom{r}{j}\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^{n-k} w_j}{\partial\tau^{n-k}}\frac{\partial^{r-j+k} b}{\partial\tau^k \partial s^{r-j}}\right] +$$

$$\frac{\partial^n}{\partial\tau^n}\sum_{j=0}^{r}\binom{r}{j}\left[\sum_{k=j+1}^{m-1}\frac{k!}{(k-j)!}w_k s^{k-j} + \frac{m!}{(m-j)!}s^{m-j}\right]\frac{\partial^{r-j} b}{\partial s^{r-j}}.$$

Hence, for $r = i$, $n = n_i$ and by evaluating at the critical point $(0,0)$ we have:

$$\left.\frac{\partial^{i+n_i} f}{\partial\tau^{n_i} \partial s^i}\right|_{(0,0)} = \sum_{j=0}^{i}\left[j!\binom{i}{j}\sum_{k=0}^{n_i}\binom{n_i}{k}\frac{\partial^{n_i-k} w_j}{\partial\tau^{n_i-k}}\frac{\partial^{i-j+k} b}{\partial\tau^k \partial s^{i-j}}\right]\Bigg|_{(0,0)}.$$

(iii) The first nonzero partial derivatives in s of f are given by:

$$\frac{\partial^m f}{\partial s^m} = \sum_{j=0}^{m-1} w_j\left[\frac{\partial^m b}{\partial s^m} + jm\sum_{k=1}^{j}\frac{\partial^{m-k} b}{\partial s^{m-k}}s^{j-k}\right] + m\sum_{k=1}^{m-1}(k+1)!\frac{\partial^{m-k} b}{\partial s^{m-k}}s^{m-k} + \frac{\partial^m b}{\partial s^m}s^m + m!b.$$

By evaluating it at the origin, it becomes $m!b(0,0)$.

□

Under some appropriate considerations, the leading terms of the Weierstrass polynomial can be derived by means of the Taylor expansion of $f(s,\tau)$. In other words, the first nonzero partial derivatives of f evaluated at$(0,0)$ can determine the first terms of the Weierstrass polynomial, we formalize this discussion through the following result.

Proposition 2.3.3. *Let $n_i < \infty$ be defined as in* (2.14)*, such that it satisfy*

$$n_0 > n_1 > \cdots > n_{m-1}.$$

Then, the coefficients $w_i(\tau)$ of the associated Weierstrass polynomial W have order $\operatorname{ord}(w_i(\tau)) = n_i$*. Moreover, the leading terms are given by*

$$w_i(\tau) = \left(\frac{m!}{i!n_i!\left.\frac{\partial^m f}{\partial s^m}\right|_{(0,0)}}\left.\frac{\partial^{i+n_i} f}{\partial\tau^{n_i}\partial s^i}\right|_{(0,0)}\right)\tau^{n_i} + o(\tau^{n_i}),\ i = 0,1,\ldots,m-1.$$

Proof. According to Theorem 1.3.2, we have that

$$f(s,\tau) = W(s,\tau)b(s,\tau),$$

where $b(0,0) \neq 0$. Now, by applying Lemma 2.3.1-(ii) to the above expression and from the fact that $n_0 > \cdots > n_i$, we deduce that

$$\left.\frac{\partial^{i+n_i} f}{\partial \tau^{n_i} \partial s^i}\right|_{(0,0)} = i! \sum_{k=0}^{n_i} \binom{n_i}{k} \left.\frac{\partial^{n_i-k} w_i}{\partial \tau^{n_i-k}} \frac{\partial^k b}{\partial \tau^k}\right|_{(0,0)} \tag{2.15}$$

$$\Rightarrow (b(0,0)i!) \left.\frac{\partial^{n_i} w_i}{\partial \tau^{n_i}}\right|_{(0,0)} = \left.\frac{\partial^{i+n_i} f}{\partial \tau^{n_i} \partial s^i}\right|_{(0,0)}. \tag{2.16}$$

Next, from (2.14) we have that $\text{ord}_\tau (w_i) \equiv n_i$ and since we know from Theorem 1.3.2 that $w_i(\tau)$ are analytic functions, this implies that

$$w_i(\tau) = w_{i,0}\tau^{n_i} + o(\tau^{n_i}). \tag{2.17}$$

Then, it is clear to see from (2.16) and (2.17) that

$$\left.\frac{\partial^{i+n_i} f}{\partial \tau^{n_i} \partial s^i}\right|_{(0,0)} = i! n_i! w_{i,0} b(0,0).$$

Finally, from Lemma 2.3.1-(iii) we get the desired result. □

2.3.3 Invariant Roots

Note that it is possible to have some $\kappa \in \mathbb{N}$ for which $n_0 = n_1 = \cdots = n_{\kappa-1} = \infty$. Then, under this situation, the Newton diagram method can not be applied directly. However, it is worth noting that since w_i are analytic functions, the previous situation is equivalent to $w_i(\tau) \equiv 0$ for $0 \leq i \leq \kappa - 1$. Hence, f will be locally given by

$$f(s,\tau) = s^\kappa \left[s^{m-\kappa} + w_{m-1}(\tau)s^{m-\kappa-1} + \cdots + w_\kappa(\tau)\right] b(s,\tau).$$

Thus, there are *κ-invariant* solutions $s = 0$ for all τ and $m - \kappa$ solutions of the form

$$s_i(\tau) = \sum_{j=1}^{\infty} c_j \tau^{j/m_i},$$

where $m_i < m$.
From this property, it can easily be seen that the invariant roots are explicit by means of a simple characterization of the partial derivatives, n_i. Hence, the effects of the delay parameter are only shown over $m - \kappa$ roots $s_i(\tau)$ in a neighborhood of (s^*, τ).

2.4 Asymptotic Root Behavior Characterization

The main goal of this subsection is to explore some qualitative properties of the solutions $s(\tau)$ of the quasi-polynomial $f(s, \tau)$ around the $m-$multiple critical pair.

2.4.1 Newton Diagram

The Newton Diagram Method is a routine for the computation of algebraic function of the form $s(\tau) = \phi(\tau^{1/n})$. In essence, a union of line segments determines the nature of leading terms of the expansion of these solutions. In this section, we give conditions to apply this method to quasi-polynomials.
The first non-zero partial derivatives, characterized by n_i (2.14), determine the Newton polygon Proposition 2.4.1. In a nutshell, n_i gives the order of the coefficients of the Weierstrass polynomial. This characterization can be obtained directly from the approximation following the numerical method in Section 2.3.1 or otherwise using the first partial derivatives labeled by n_i described in Section 2.3.2. The following proposition formalizes this discussion.

Proposition 2.4.1. *Let $s = 0$ be an m-multiple root at $\tau = 0$ of the quasi-polynomial $f(s, \tau)$, and assume that $n_0 < \infty$. Then, the Newton diagram of f at $(0, 0)$ is given by $\Pi = \{(0, n_0), \ldots, (m-1, n_{m-1}), (m, 0)\}$.*

Proof. First note from the Weierstrass Preparation Theorem, the definition of n_i and Lemma 2.3.1, that the Newton diagram of f has the end points at $(0, n_0)$ and $(m, 0)$. Moreover, around the singular point $(0, 0)$, f can be written as

$$f(s, \tau) = \sum_{i=0}^{\infty} \left\{ \frac{1}{i!} \sum_{j=i}^{\infty} \binom{j}{i} \frac{\partial^j f}{\partial s^i \partial \tau^{j-i}} \tau^{j-i} \right\} s^i \tag{2.18}$$

or equivalently as,

$$f(s,\tau)=\sum_{i=0}^{m-1}\left\{\frac{1}{i!}\sum_{j=n_i+1}^{\infty}\binom{j}{i}\frac{\partial^j f}{\partial s^i\partial\tau^{j-i}}\tau^{j-i}\right\}s^i+\sum_{i=m}^{\infty}\left\{\frac{1}{i!}\sum_{j=i}^{\infty}\binom{j}{i}\frac{\partial^j f}{\partial s^i\partial\tau^{j-i}}\tau^{j-i}\right\}s^i.$$

Hence, from Lemma 2.3.1-(ii), the remaining $(m-2)-$points of the Newton diagram of f are given by the set $\{(1,n_1)\ldots,(m-1,n_{m-1})\}$, which concludes the proof. □

Remark 2.4.1. *If $\kappa\in\mathbb{N}$ for which $n_0=n_1=\cdots=n_{\kappa-1}=\infty$, under this consideration the Newton polygon will be given by $\Pi=\{(\kappa,n_\kappa),\ldots,(m,0)\}$. If such number κ does not exist (i.e., if such situation does not happen), then κ will be simply defined as $\kappa:=0$.*

Example 2.4.1. *In order to illustrate the previous result, let us consider the following quasi-polynomial (borrowed from [20]):*

$$f(s,\tau)=-\left(\frac{\pi}{2}s^5+\frac{\pi}{2}s^3+s^2\right)+\left(\frac{\pi}{2}s^3-s^2+\frac{\pi}{2}s+1\right)e^{-s\tau}+e^{-2s\tau},\qquad(2.19)$$

with $s=\boldsymbol{i}$ a multiple root at $\tau=\pi$ of multiplicity $m=3$. First, let us derive the constants n_i considered in (2.14):

$$\left.\frac{\partial f}{\partial\tau}\right|_{(\boldsymbol{i},\pi)}=0,\ \left.\frac{\partial^2 f}{\partial\tau^2}\right|_{(\boldsymbol{i},\pi)}=-2\ \Rightarrow n_0=2,\quad \left.\frac{\partial^2 f}{\partial s\partial\tau}\right|_{(\boldsymbol{i},\pi)}=2+\boldsymbol{i}\pi,\Rightarrow n_1=1$$

$$\left.\frac{\partial^3 f}{\partial s^2\partial\tau}\right|_{(\boldsymbol{i},\pi)}=-\left(5\pi+\boldsymbol{i}(4\pi^2+6)\right)\Rightarrow n_2=1,\quad \left.\frac{\partial^3 f}{\partial s^3}\right|_{(\boldsymbol{i},\pi)}=-3\pi(-6-5\boldsymbol{i}\pi+\pi^2).$$

Summarizing, we have $(n_0,n_1,n_2)=(2,1,1)$. Thus, according to Proposition 2.4.1, we have $\Pi=\{(0,2),\ (1,1),(2,1),(3,0)\}$. Such points are depicted in Fig. 2.1.

From the above, it can be seen that this method requires a graphic construction, with the aim of computing the first term in the expansion of the solution. Along these lines, based on the Newton procedure introduced in Section 1.4, using as input n_i $(0\leq n_i\leq m)$ the following algorithm Algorithm 1 gives as output, the

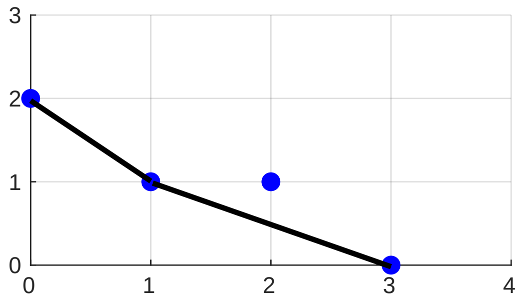


Figure 2.1: Newton Diagram of the quasi-polynomial (2.19), determined by the set of points $\Pi = \{(0,2),(1,1),(2,1),(3,0)\}$.

segments Π^i of the Newton polygon and the exponents β_i of the solution terms.

Algorithm 1: Auxiliary Puiseux Series Expansion

Result: Newton Polygon of $f(s,\tau)$, with an m-multiple root $s^* = \boldsymbol{i}\omega^*$ at $\tau = \tau^*$.

Input : Initial values as $r := 0$, $i_{-1} := \kappa$ and $\ell_{-1} := n_\kappa$.

Output: Obtaining segments with points Π^j, slope $-\beta_j$ and partial multiplicity m_j.

while $i_{r-1} < m$ **do**

  set $\mathcal{E}_r := \left\{ \frac{\ell - \ell_{r-1}}{i_{r-1} - i} : (i,\ell) \in \Pi, \text{ and } i > i_{r-1} \right\}$

  $\beta_r := \max \mathcal{E}_r$ and $\Pi^{(r)} := \left\{ (i,\ell) \in \Pi : \beta_r \equiv \frac{\ell - \ell_{r-1}}{i_{r-1} - i} \right\} \bigcup \{(i_{r-1}, \ell_{r-1})\}$

  set $(i_r, \ell_r) \in \Pi^{(r)}$ such that $i_r \geq i, \ \forall (i,\ell) \in \Pi^{(r)}$

  set $m_r := i_r - i_{r-1}$ and $r = r + 1$

end

The above algorithm will be shown to be useful in the development of the next result.

Example 2.4.2. *With the aim of illustrating the simplicity of the algorithm, let us apply it to the quasi-polynomial (Example 2.4.1 above). First, observe that according to Remark 2.4.3 we have that* $\kappa = 0$, *implying that the initial conditions for the Algorithm 1 are* $(r, i_{-1}, \ell_{-1}) = (0,0,2)$. *The Newton Diagram is given by the set* $\Pi = \{(0,2),(1,1),(2,1),(3,0)\}$, *determined in Example 2.4.1, and depicted in Fig. 2.1. Then, the algorithm find the first segment and its slope* β_0. *Thus, for the first iteration we have that* $\mathcal{E}_0 = \left\{1, \frac{1}{2}, \frac{2}{3}\right\} \Rightarrow \beta_0 = 1$ *and* $\Pi^{(0)} = \{(0,2),\ (1,1)\}$.

According to step 3 we have $(i_0, \ell_0) = (1, 1)$, *which implies that the algorithm will end in the next iteration. The following table summarizes the results:*

Table 2.1: Results summary for the quasi-polynomial (2.19).

Initial Data	Algorithm Output
$m = 3$, $n_0 = 2$ $\Pi = \{(0, 2), (1, 1), (2, 1), (3, 0)\}$	$r = 1$, $m_0 = 1$, $\beta_0 = 1$ $\Pi^{(0)} = \{(0, 2), (1, 1)\}$
	$r = 2$, $m_1 = 2$, $\beta_1 = \frac{1}{2}$ $\Pi^{(1)} = \{(1, 1), (3, 0)\}$

2.4.2 Splitting Properties

As discussed by [90], it is possible to characterize the root locus of f by its branches. In fact, the equation $f(s, \tau) = 0$ defines a solution curve $\mathcal{C} \in \mathbb{C}^2$ which is composed by the finite union of $r-$branches $s_j\left(\tau^{1/m_j}\right)$, each of these branches can be expressed as a Puiseux series:

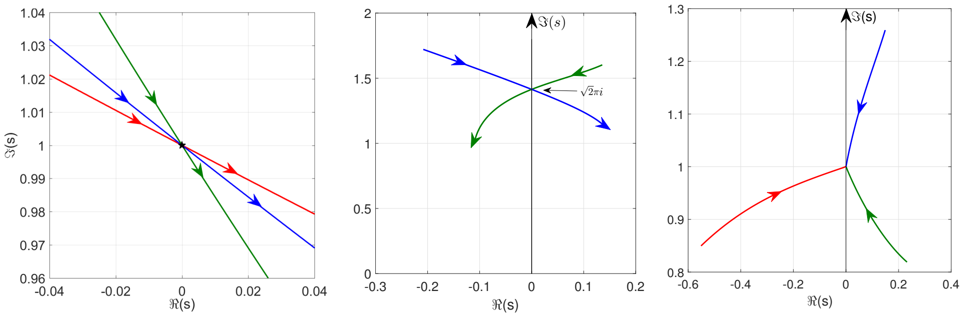
$$s_{j\sigma}(\tau) = c_{j\sigma}\tau^{\frac{1}{m_j}} + o\left(|\tau|^{\frac{1}{m_j}}\right), \quad j = 0, \ldots, r - 1, \quad \sigma = 1, \ldots, m_j,$$

where each branch has multiplicity m_j, such that $m = m_1 + m_2 + \cdots + m_r$. In the case when $r = 1$, then $s_{j\sigma}$ and $c_{j\sigma}$ will be simply denoted by s_σ and c_σ, respectively.

Definition 2.4.1. *We say that there is a* Complete Regular Splitting *(CRS) property of the solution* $s^* = 0$ *at* $\tau^* = 0$ *if* $c_{j\sigma} \neq 0$, $\forall j$. *For the* Regular Splitting *(RS) property, some of the coefficients* $c_{j\sigma}$ *for which* $m_j = 1$ *may be equal to zero. In the remaining cases of the coefficient* $c_{j\sigma}$ *we say that Non Regular Splitting property is present.*

Remark 2.4.2. *The above definition, illustrated in Fig. 2.2, was inspired by the matrix case introduced in [47] (see also [38]).*

The proposed approach to deal with the splitting properties is based on the Newton diagram method applied in conjunction with the Weierstrass polynomial and the Puiseux Theorem. To this end, we will use the definitions introduced in the previous sections; in particular the notion of Newton polygon Π.



(a) Non Regular Splitting (b) Regular Splitting (c) Complete Regular Splitting

Figure 2.2: Splitting properties.

Proposition 2.4.2. *Let* $s^* = \boldsymbol{i}\omega^*$ *at* $\tau = \tau^*$ *be a* m*-multiple critical root of the quasi-polynomial* $f(s, \tau)$*. Assume that* r, β_j, (i_j, ℓ_j), m_j *and* $\Pi^{(j)}$, *for* $j = 0, 1, \ldots, r-1$ *are given by the Algorithm 1. Then, the following properties hold:*

(i) *if* $m_j \beta_j = 1$, $\forall j \in \{0, 1, \ldots, r-1\}$, *then the solution* $(\boldsymbol{i}\omega^*, \tau^*)$ *of* $f(s, \tau)$ *has the completely regular splitting property;*

(ii) *if some* β_j *satisfies* $m_j \cdot \beta_j > 1$ *for* $m_j > 1$, *then non regular splitting property for the solution* $(\boldsymbol{i}\omega^*, \tau^*)$ *occur;*

(iii) *if the pairs* (m_k, β_k) *that do not fulfill (i), satisfy the inequality* $\beta_k \geq m_k \equiv 1$, *then the solution* $(\boldsymbol{i}\omega^*, \tau^*)$ *of* $f(s, \tau)$ *has the regular splitting property;*

(iv) *let* Ω_0 *be a neighborhood of* $(0, 0) \in \mathbb{C}^2$, *and assume that* $\mathcal{R}\left(W, \frac{\partial}{\partial s} W\right) \neq 0$, $\forall s \in \Omega_0 \setminus \{(0, 0)\}$. *Then, there are* $m-$*different Puiseux series solutions* $g_i\left(\tau^{\frac{1}{n_i}}\right)$ *such that,*

$$f(s, \tau) = \prod_{i=1}^{m} \left(s - g_i\left(\tau^{\frac{1}{n_i}}\right)\right) b(s, \tau),$$

where n_i *is arranged in terms of* m_j *as*

$$\underbrace{n_1 = n_2 = \cdots = n_{m_1}}_{n_{i_1} = m_1}, \underbrace{n_{m_1+1} = \cdots = n_{m_2}}_{n_{i_2} = m_2}, \cdots, \underbrace{n_{m_1+\cdots+m_{r-1}+1} = \cdots = n_{m_1+\cdots+m_r}}_{n_{i_r} = m_r}$$

with $\sum m_i = m$.

Proof. First of all, observe that r in the Algorithm 1 corresponds to the number of branches for the solution $(\boldsymbol{i}\omega^*, \tau^*)$ and m_j the multiplicity associated to each branch.

(i) In this case, we have that $\beta_j = \frac{1}{m_j}$, then from the Newton procedure we know that the rational numbers β_j are associated to the first exponents in the solutions, since we have r branches, thus the root locus of $f(s, \tau)$, is given by

$$s_{j\sigma}(\tau) = c_{j\sigma}\tau^{\frac{1}{m_j}} + o\left(\tau^{\frac{1}{m_j}}\right) \quad j = 0, \ldots, r-1 \quad \sigma = 1, \ldots, m_j.$$

Since $c_{j\sigma}$ are related to the nonzero solution of a polynomial formed with the coefficients of the convex hull, clearly $c_{j\sigma} \neq 0$. Then, the solution $(\boldsymbol{i}\omega^*, \tau^*)$ has the CRS property.

(ii)-(iii) These follow in similar lines than those presented in (i).

(iv) (See [21]). This case can be stated by induction. To this end, from the Puiseux Theorem we know that there exists a Puiseux series $g_1\left(\tau^{1/n_i}\right)$ such that $f(g_1, \tau) = 0$. Then, the solution $g_1(\tau^{1/n_i})$ can be factored, such that f is written as the product $f(s, \tau) = g_1(\tau)f_1(s, \tau)$, where $f_1 \in \mathbb{C}\{s, \tau\}$. Assume now that the above factorization is valid for some $k \in \mathbb{N}$, i.e., the following relation holds:

$$f(s, \tau) = g_1(\tau) \cdots g_k(\tau)f_k(s, \tau),$$

Then the quotient f_k has order $m-k$ in s. Applying the induction hypothesis to f_k, then we get m different factors g_i such that:

$$f(s, \tau) = g_1(\tau) \cdots g_m(\tau)f_m(s, \tau),$$

where $f_m(s, \tau)$ has order $\text{ord}_\tau\left(f_m\right) = 0$.

□

Corollary 2.4.1. *Consider the same hypothesis as in Proposition 2.4.2. Assume that* $n_0 = 1$. *Then at* $\tau = \tau^*$ *the* $m-$*roots of* $f(s,\tau)$ *have the CRS property, i.e. these roots can be expanded as*

$$s_\sigma(\tau) = \boldsymbol{i}\omega^* + c_\sigma (\tau - \tau^*)^{\frac{1}{m}} + o\left(|\tau - \tau^*|^{\frac{1}{m}}\right), \quad \text{for } \sigma = 1, 2, \ldots, m. \tag{2.20}$$

Moreover, the following properties hold:

(i) *if* $m = 2$ *and* $\Re(c_\sigma) \neq 0$ *with* $\sigma \in \{1,2\}$. *Then for* $\tau > \tau^*$ *sufficiently close to* τ^*, *one of the zeros* $s_\sigma(\tau)$ *will enter the RHP, whereas the other one will enter the LHP;*

(ii) *if* $m > 2$, *then at least one of the zeros* $s_\sigma(\tau)$ *will enter the RHP.*

2.4.3 Crossing Directions Characterization

As mentioned in previous sections, the Weierstrass polynomial will be the main tool to analyze the stability behavior of the critical characteristic roots. In the same spirit as [24], the following result is obtained:

Proposition 2.4.3. *Let* $n_0 < \infty$ *and* $s^* = \boldsymbol{i}\omega^*$ *be an* $m-$*multiple root of* $f(s,\tau)$ *at* $\tau = \tau^*$. *Assume that* r, β_j, (i_j, ℓ_j), m_j *and* $\Pi^{(j)}$, *for* $j = 0, 1, \ldots, r-1$ *are given by the Algorithm 1. Then, at* $\tau = \tau^*$, *the* $m-$*zeros of* $f(s,\tau)$ *can be expanded as*

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_{j\sigma} (\tau - \tau^*)^{\beta_j} + o\left(|\tau - \tau^*|^{\beta_j}\right), \tag{2.21}$$

for $j = 0, 1, \ldots, r-1$, $\sigma = 1, \ldots, m_j$ *and* $m = m_0 + \cdots + m_{r-1}$, *where* $c_{j\sigma}$ *are roots of the polynomial* $\mathcal{P}_j : \mathbb{C} \mapsto \mathbb{C}$,

$$\mathcal{P}_j(z) := \sum_{k=i_{j-1}}^{i_j} a_{k,0} z^{k-i_{j-1}}, \quad s.t. \ (k, \eta_k) \in \Pi^{(j)}, \tag{2.22}$$

where the coefficients $a_{k,0} \in \mathbb{C}$ *is given by*

$$a_{k,0} = \left(\frac{m!}{k!\eta_k! \left.\frac{\partial^m f}{\partial s^m}\right|_{(0,0)}} \right) \left.\frac{\partial^{k+\eta_k} f}{\partial \tau^{\eta_k} \partial s^k}\right|_{(0,0)}. \tag{2.23}$$

Furthermore, for $\tau > \tau^*$ *sufficiently close to* τ^*, *the zeros* $s_{j\sigma}(\tau)$ *will enter the right half-plane (or to the left half-plane) if*

$$\Re\{c_{j\sigma}\} > 0 (< 0). \tag{2.24}$$

Proof. By taking into account the Newton procedure and applying the Algorithm 1 to $f(s,\tau)$, it is clear to see that the solution $s_{j\sigma}$ can be expanded as in (2.21). Now, observe that proving that the coefficients $c_{j\sigma}$ are given by the solutions of $\mathcal{P}_j$ is equivalent to prove that the coefficients of the Weierstrass polynomial W that lies on the Newton polygon are given by (2.23) modulus some constant factor. Thus, in order to show this fact we first notice that, according to Lemma 2.3.1-(i), $\operatorname{ord}_\tau(w_0) = n_0$. Moreover, from the definition of n_i, it is clear to see that $\operatorname{ord}_\tau(w_i) = n_i$. Next, since by Theorem 1.3.2, we have that $f = Wb$ with $b(0,0) \neq 0$, and according to Lemma 2.3.1-(ii), we know that the first ℓ derivatives in s and the $\tilde{n}$ derivatives in τ are given by:

$$\frac{\partial^{\ell+\tilde{n}} f}{\partial\tau^{\tilde{n}}\partial s^{\ell}} = \sum_{j=0}^{\ell}\left[j!\binom{\ell}{j}\sum_{k=0}^{\tilde{n}}\binom{\tilde{n}}{k}\frac{\partial^{\tilde{n}-k} w_j}{\partial\tau^{\tilde{n}-k}}\frac{\partial^{\ell-j+k} b}{\partial\tau^{k}\partial s^{\ell-j}}\right] + \tag{2.25}$$

$$\frac{\partial^{\tilde{n}}}{\partial\tau^{\tilde{n}}}\sum_{j=0}^{\ell}\binom{\ell}{j}\left[\sum_{k=j+1}^{m-1}\frac{k!}{(k-j)!}w_k s^{k-j} + \frac{m!}{(m-j)!}s^{m-j}\right]\frac{\partial^{\ell-j} b}{\partial s^{\ell-j}}. \tag{2.26}$$

Let $\upsilon_j, \alpha_{\upsilon_j} \in \mathbb{N}\cup 0$ such that $(\upsilon_j, \alpha_{\upsilon_j}) \in \Pi^{(j)}$, then clearly these points will lie on the Newton polygon. Furthermore, such points satisfy the relation $\alpha_{\upsilon_j} > \alpha_{\upsilon_k}$ for $\upsilon_j < \upsilon_k$. Therefore, by taking $\ell = \upsilon_j$ and $\tilde{n} = \alpha_{\upsilon_j}$ in (2.26) it is clear to see from (2.16) and (2.17) that

$$\left.\frac{\partial^{\upsilon_j+\alpha_{\upsilon_j}} f}{\partial\tau^{\alpha_{\upsilon_j}}\partial s^{\upsilon_j}}\right|_{(0,0)} = \upsilon_j!\alpha_{\upsilon_j}!w_{\upsilon_j,0}b(0,0).$$

Then, (2.23) follows by noting that $k = \upsilon_j$ and $\eta_k = \alpha_{\upsilon_j}$. Finally, the direction of crossing follows straightforwardly by condition (2.24). □

Remark 2.4.3. *It is worth mentioning that in the above result, it is not necessary to assume that $n_0 < \infty$ (i.e., $\kappa \neq 0$). In fact, in order to relax such an assumption, i.e., to consider $\kappa > 0$, we have just assume that the $j-$index will take values in the set $\{\kappa, \kappa+1, \ldots, r-1\}$ and that the solution $s_{j\sigma}$ will be arranged in $(r-\kappa)-$branches of Puiseux series.*

2.4.4 Higher-Order Analysis

In some situations, the first-order expansion does not give enough information to analyze the stability of a given solution. Such situations occur when (2.24) does not hold, that is when the coefficient of the first-order term is purely imaginary. Thus, in order to cope with such a case study, and inspired by the results developed by [89], in the following, we will consider a higher-order analysis.

Let β_j, $(i_j, \ell_j) \in \Pi^{(j)}$ be given by the Algorithm 1. It is not difficult to see that, in order to compute higher-order terms for the solution $s_{j\sigma}(\tau)$, we can make use of the change of variables $s \mapsto \tau^{\beta_j}(c_{j\sigma} + s_1)$ in $f(s,\tau)$ to get the function $f_1(s_1, \tau)$, and repeat the same procedure presented in the previous section for this function f_1. In this vein, the solution $s_{j\sigma}$ can be expressed as

$$s_{j\sigma}(\tau) = c_1\tau^{\beta_j} + c_2\tau^{\beta_j+\beta_j^{(1)}} + c_3\tau^{\beta_j+\beta_j^{(1)}+\beta_j^{(2)}} + \cdots , \tag{2.27}$$

where $\beta_j^{(1)}$ is the output of the Algorithm 1 for the function f_1, and so on. Now, according to the Newton procedure, we have that any arbitrary pair (i, η_i) belonging to the set $\Pi^{(j)}$ must satisfy $\eta_i + i\beta_j = \nu_j$, with a fixed $\nu_j \in \mathbb{Q}$. Hence, in order to find some insights over c_2 and $\beta_j^{(1)}$, let us define the associated Weierstrass polynomial W_1 as $W_1(s_1, \tau) := \tau^{-\nu_j} W(\tau^{\beta_j}(c_{j\sigma} + s_1), \tau)$. Then, from all these facts, W_1 will be given as

$$W_1\left(s_1,\tau\right) = \tau^{-\nu_j}\left[\tau^{m\beta_j}\left(c_1+s_1\right)^m + w_{m-1}(\tau)\tau^{(m-1)\beta_j}\left(c_1+s_1\right)^{(m-1)} + \cdots + w_0(\tau)\right].$$

Now, since for the first-order term of the solution $s_{j\sigma}$, we only need to consider the terms on $\Pi^{(j)}$, according to Proposition 2.4.3, we have that the main coefficients of W falling on $\Pi^{(j)}$ are denoted by $a_{l,0}$, therefore W_1 can be rewritten as

$$\begin{aligned} W_1(s_1,\tau) = {} & \underbrace{\tau^{-\nu_j}\sum_{l=i_{j-1}}^{i_j} a_{l,0}\tau^{\eta_l + l\beta_j}\left(c_1+s_1\right)^l +}_{=:W_1^{(m)}(s_1,\tau)} \\ & \underbrace{+\,\tau^{-\nu_j}\left[\sum_{l=i_{j-1}}^{i_j}\left(w_l - a_{l,0}\tau^{\eta_l}\right)\tau^{l\beta_j}\left(c_1+s_1\right)^l + \sum_{h\notin\Pi^{(j)}} w_h\tau^{h\beta_j}\left(c_1+s_1\right)^h\right]}_{=:W_1^{(r)}(s_1,\tau)}. \end{aligned} \tag{2.28}$$

Taking into consideration that $\eta_l + l\beta_j = \nu_j$, the first summation in (2.28) can be reduced as follows

$$\sum_{l=i_{j-1}}^{i_j} a_{l,0}\tau^{n_l+l\beta_j}\left(c_1+s_1\right)^l = \tau^{\nu_j}\left(c_1+s_1\right)^{i_{j-1}}\mathcal{P}_j\left(c_1+s_1\right),$$

$$= \tau^{\nu_j}\left(c_1+s_1\right)^{i_{j-1}} s_1^{\mu}\psi\left(c_1+s_1\right), \tag{2.29}$$

where, according to Proposition 2.4.3, c_1 is in general a $\mu-$multiple solution of $\mathcal{P}_j$ and $\psi(c_1) \neq 0$. Thus, $W_1^{(m)}$ will be expressed as

$$W_1^{(m)}(s_1,\tau) = c_1^{i_{j-1}}\psi(c_1)s_1^{\mu} + \left(i_{j-1}c_1^{i_{j-1}-1}\psi(c_1) + c_1^{i_{j-1}}\psi'(c_1)\right)s_1^{\mu+1} + \cdots .$$

From the above discussion, it is clear to see that $\operatorname{ord}_\tau\left(W_1^{(m)}\right) \equiv 0$. Now, since $c_1^{i_{j-1}}\psi(c_1) = constant \neq 0$, we have that W_1 can be expressed as

$$W_1(s_1,\tau) = w_0^{(1)}(\tau) + \cdots + w_\mu^{(1)}(\tau)s_1^{\mu} + \cdots + w_m^{(1)}(\tau)s_1^{m}, \tag{2.30}$$

with $\operatorname{ord}_\tau\left(w_\mu^{(1)}\right) = 0$, implying that the end point for the Newton polygon of W_1 will be $(\mu, 0)$. Bearing in mind these facts, we have the following:

Proposition 2.4.4. *Let $s^* = \boldsymbol{i}\omega^*$ be an $m-$multiple root of $f(s,\tau)$ at $\tau = \tau^*$. Assume that β_j and m_j for $j = \kappa, \kappa+1, \ldots, r-1$ are given by the Algorithm 1. If $\beta_j = 1$, then the following statements hold:*

(i) the equation $f(s,\tau) = 0$ has m_j-solutions of the form

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_{j\sigma}\left(\tau-\tau^*\right) + o\left(|\tau-\tau^*|\right), \quad \sigma = 1,\ldots,m_j, \tag{2.31}$$

where $c_{j\sigma}$ is a root of the polynomial $\mathcal{P}_j$ defined in (2.22);

(ii) if $c_{j\sigma}$ is a simple root of $\mathcal{P}_j$ then, there are m_j-solutions expanded as a Taylor series in the form

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_{j\sigma}\left(\tau-\tau^*\right) + c_{j\sigma}^{(1)}\left(\tau-\tau^*\right)^{1+\beta_j^{(1)}} + \cdots ,$$

where $\beta_j^{(1)} \in \mathbb{N}$.

Proof. Let m_j and β_j be given by the Algorithm 1.

(i) Since by hypothesis $\beta_j = 1$, the condition (2.31) follows straightforwardly from the Newton procedure.

(ii) Next, by hypothesis, we have that $c_{j\sigma}$ is a simple root of $\mathcal{P}_j$. Thus, using the same arguments as in the previous discussion, we have that $\mu \equiv 1$ implies that (2.30) can be written as

$$W_1(s_1, \tau) = w_0^{(1)}(\tau) + w_1^{(1)}(\tau)s_1 + \cdots + w_m^{(1)}(\tau)s_1^m,$$

where $\mathrm{ord}_\tau \left(w_1^{(1)}\right) = 0$. Now, since:

$$w_0^{(1)}(\tau) = \tau^{-\nu_j} \left(w_0(\tau) + c_{j\sigma}w_1(\tau)\tau + \cdots + c_{j\sigma}^{m-1}w_{m-1}(\tau)\tau^{m-1} + c_{j\sigma}^m\tau^m\right),$$

we have that $w_0^{(1)}(0) \equiv 0 \Rightarrow \mathrm{ord}_\tau \left(w_0^{(1)}\right) \geq 1$. Thus, since according to Theorem 1.3.2 we know that $w_i^{(1)}$ are analytic functions, and since $\mathrm{ord}_\tau \left(w_0^{(1)}\right) \geq 1$ one gets $\beta_j^{(1)} \in \mathbb{N}$.

□

Proposition 2.4.5. *Let $s^* = \boldsymbol{i}\omega^*$ be an $m-$multiple root of $f(s,\tau)$ at $\tau = \tau^*$. Assume that β_j, m_j and $(i_j, \ell_j) \in \Pi^{(j)}$ for $j = 0, 1, \ldots, r-\kappa-1$ are given by the Algorithm 1. If $\beta_j = 1/m_j$, then $f(s,\tau) = 0$ has m_j-solutions given by*

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_j\Theta_\sigma\left(\tau - \tau^*\right)^{1/m_j} + o(|\tau - \tau^*|^{1/m_j}), \quad \sigma = 1, \ldots, m_j,$$

where $\Theta_\sigma = \exp\left(\boldsymbol{i}\frac{\theta_j + 2\pi(\sigma-1)}{m_j}\right)$, $\theta_j = \arg(c_j^{m_j})$ and $c_j = \left|a_{i_{j-1},0}/a_{i_j,0}\right|^{1/m_j}$.

Proof. The proof follows straightforwardly from Proposition 2.4.2. □

Proposition 2.4.6. *Let $s^* = \boldsymbol{i}\omega^*$ be an $m-$multiple root of $f(s,\tau)$ at $\tau = \tau^*$. Let β_j, m_j and $(i_j, \ell_j) \in \Pi^{(j)}$ for $j = 0, 1, \ldots, r-\kappa-1$ be given by the Algorithm 1. Assume that $\beta_j = 1$, $c_{j\sigma}$ is an m_j-multiple root of $\mathcal{P}_j$ and $\frac{d^{\nu_j+1}}{\tau^{\nu_j+1}} f\left(c_{j\sigma}\tau, \tau\right)\Big|_{\tau=0} \neq 0$, with $\nu_j = n_{i_{j-1}} + i_{j-1}$. Then, there are m_j-solutions expanded as a Puiseux Series in the form*

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_{j\sigma}\left(\tau - \tau^*\right) + c_{j\sigma}^{(1)}\left(\tau - \tau^*\right)^{1+1/m_j} + o\left(|\tau - \tau^*|^{1+1/m_j}\right), \quad \sigma = 1, \ldots, m_j,$$

where $c_{j\sigma}^{(1)}$ is a solution of the polynomial $\mathcal{P}_j$ given in (2.22), associated to W_1.

Proof. Let m_j, β_j and (i_j, ℓ_j) be given by the Algorithm 1. First, note that since $\beta_j \equiv 1$ clearly the m_j-solution will be expanded as

$$s_{j\sigma}(\tau) = \boldsymbol{i}\omega^* + c_{j\sigma}(\tau - \tau^*) + o(|\tau - \tau^*|), \quad \sigma = 1, \ldots, m_j.$$

Now, using similar arguments to the previous case study we have that $\mu \equiv m_j$, implies that (2.30) can be written as

$$W_1(s_1, \tau) = w_0^{(1)}(\tau) + \cdots + w_{m_j}^{(1)}(\tau)s_1^{m_j} + \cdots + w_m^{(1)}(\tau)s_1^m,$$

where $\mathrm{ord}_\tau\left(w_{m_j}^{(1)}\right) = 0$. Thus, in order to have $\beta_j^{(1)} = 1/m_j$, $\mathrm{ord}_\tau\left(w_0^{(1)}\right) = 1$ must be fulfilled. To see that such a condition hold, note that $w_0^{(1)}(\tau) = \tau^{-\nu_j} W(c_{j\sigma}\tau, \tau)$, and by Theorem 1.3.2, we have that $f = Wb$, where $\mathrm{ord}(b) = 0 \Rightarrow \mathrm{ord}_\tau(Wb) = \mathrm{ord}_\tau(W)$. Since

$$\mathrm{ord}_\tau\left(w_0^{(1)}\right) = 1 \Leftrightarrow \left.\frac{\partial}{\partial\tau} w_0^{(1)}\right|_{\tau=0} \neq 0,$$

we have that

$$\left.\frac{\partial}{\partial\tau} w_0^{(1)}\right|_{\tau 0} \neq 0 \Leftrightarrow \left.\frac{\partial}{\partial\tau}\left(\tau^{-\nu_j} f(c_{j\sigma}\tau, \tau)\right)\right|_{\tau=0} \neq 0.$$

Finally, by noticing that

$$\left.\frac{\partial}{\partial\tau}\left(\tau^{-\nu_j} f(c_{j\sigma}\tau, \tau)\right)\right|_{\tau=0} \neq 0 \Leftrightarrow \left.\left(\tau^{-\nu_j} f' - \nu_j \tau^{-\nu_j - 1} f\right)\right|_{\tau=0} \neq 0,$$

and since,

$$\left.\left(\tau^{-\nu_j} f' - \nu_j \tau^{-\nu_j} f\right)\right|_{\tau=0} \neq 0 \Leftrightarrow \left.\frac{d^{\nu_j+1}}{\tau^{\nu_j+1}} f(c_{j\sigma}\tau, \tau)\right|_{\tau=0} \neq 0,$$

the proof is completed. □

Remark 2.4.4. *Since the Weierstrass polynomial W is derived using the quasi-polynomial f, thus, instead of considering W_1 in Proposition 2.4.6, it is possible to consider* $f_1(s_1, \tau) := \tau^{-\nu_j} f\left(\tau^{\beta_j}(c_{j\sigma} + s_1), \tau\right)$.

2.5 Illustrative Examples

In order to illustrate the effectiveness of the proposed methodology, in the sequel several examples are proposed. The numerical computation has been performed using the software package DDE-BIFTOOL (see, for instance, [31, 30]).

Example 2.5.1. *In this section, the asymptotic behavior characterization of Example 2.4.1 is given. Remember that the quasi-polynomial is given by*

$$f(s,\tau) = -\left(\frac{\pi}{2}s^5 + \frac{\pi}{2}s^3 + s^2\right) + \left(\frac{\pi}{2}s^3 - s^2 + \frac{\pi}{2}s + 1\right)e^{-s\tau} + e^{-2s\tau}.$$

The Newton Diagram Π *and initial conditions* (r, i_{-1}, ℓ_{-1}) *for the Algorithm 1 are given in Table 2.2.*
The Newton Diagram Algorithm is described as follows, where each step of Algorithm 1 is interpreted.

1. *Line segments starting at* (κ, ρ_κ) *are defined by means of all slopes* $(\ell - \ell_{r-1})/(i_{r-1} - i)$*;*

2. *The first line segment is characterized by its slope* $\beta_r = \max \mathcal{E}_r$*, and by the points* π_k *on* $\Pi^{(r)}$*;*

3. *The endpoint of the segment is set as* $(i_r, \ell_r) \in \Pi^{(r)}$*;*

4. m_r *is defined as the projection of the line segment;*

5. *If* $i_{r-1} = m$ *the algorithm ends.*

Table 2.1 summarizes the results obtained by applying the Algorithm 1, Proposition 2.4.1 and Proposition 2.4.3. Following the considerations above, the solutions of the quasi-polynomial (2.19) *around the critical point* $(\boldsymbol{i}, \pi)$*, split into two branches:*

Table 2.2: Results summary for the quasi-polynomial (2.19).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}_j(z) = 0\}$
$m = 3$, $n_0 = 2$	$r = 0$, $m_0 = 1$, $\beta_0 = 1$	$\mathcal{P}_0(z) := -\frac{4+2\boldsymbol{i}\pi}{\pi(-6-5\boldsymbol{i}\pi+\pi^2)}z + \frac{2}{\pi(-6-5\boldsymbol{i}\pi+\pi^2)}$
$\Pi = \{(0,2), (1,1), (2,1), (3,0)\}$	$\Pi^{(0)} = \{(0,2), (1,1)\}$	$\left\{c_{0,1} = \frac{1}{2+\boldsymbol{i}\pi}\right\}$
	$r = 1$, $m_1 = 2$, $\beta_1 = \frac{1}{2}$	$\mathcal{P}_1(z) := z^2 - \frac{4+2\boldsymbol{i}\pi}{\pi(-6-5\boldsymbol{i}\pi+\pi^2)}$
	$\Pi^{(1)} = \{(1,1), (3,0)\}$	$\left\{c_{1,\ell} = -(-1)^\ell \frac{1+\boldsymbol{i}}{\sqrt{\pi(\pi-3\boldsymbol{i})}}\right\}$

$$s_{0,1}(\tau) = \boldsymbol{i} + \frac{1}{2+\boldsymbol{i}\pi}(\tau - \pi) + o(\tau - \pi),$$
$$s_{1,\ell}(\tau) = \boldsymbol{i} - (-1)^\ell \frac{1+i}{\sqrt{\pi(\pi - 3\boldsymbol{i})}}(\tau - \pi)^{1/2} + o\left((\tau-\pi)^{1/2}\right), \quad \ell = 1, 2.$$

Finally, since for the solution $s^* = \boldsymbol{i}$ *at* $\tau^* = \pi$ *we have that* $\beta_0 = 1/m_0$ *and* $\beta_1 = 1/m_1$*, according to Proposition 2.4.2, such a solution has the CRS property. Furthermore, since the solutions are given in two branches with multiplicities* m_0

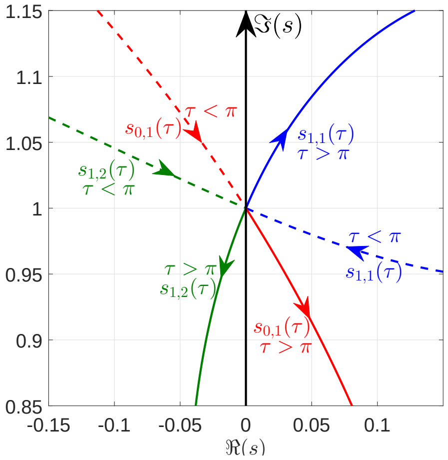


Figure 2.3: Root locus behavior, in a degenerate case, for quasi-polynomial (2.19).

and m_1*, thus by Propositions 2.4.4 and 2.4.5, one root will behave as a Taylor series (*$s_{0,1}$*, whereas the other branch will behave as a Puiseux series (*$s_{1,1}$ *and* $s_{1,2}$ *in Fig. 2.3.*

Example 2.5.2. *Consider the following quasi-polynomial (borrowed from [43]):*

$$f(s,\tau) = (s^4 + 2s^2 + 2) + 2e^{-s\tau} + e^{-2s\tau}, \tag{2.32}$$

with a critical point at $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$*, and multiplicity* $m = 2$*. Now, in order to determine the constants* n_i (2.14) *the partial derivatives are computed:*

$$\left.\frac{\partial f}{\partial \tau}\right|_{(\boldsymbol{i},\pi)} = 0, \quad \left.\frac{\partial^2 f}{\partial \tau^2}\right|_{(\boldsymbol{i},\pi)} = -2 \quad \Rightarrow n_0 = 2,$$
$$\left.\frac{\partial f}{\partial s}\right|_{(\boldsymbol{i},\pi)} = 0, \quad \left.\frac{\partial^2 f}{\partial \tau \partial s}\right|_{(\boldsymbol{i},\pi)} = 2\boldsymbol{i}\pi \Rightarrow n_1 = 1,$$

and since $n_0 < \infty$*, this implies that* $\kappa = 0$*. Then, the points on the Newton polygon for the quasi-polynomial* f *at the critical point* $(\boldsymbol{i}, \pi)$ *are given by the following set:*

$$\Pi = \{(0, n_0), (1, n_1), (2, 0)\} = \{(0, 2), (1, 1), (2, 0)\}.$$

Table 2.3: Results summary for the quasi-polynomial (2.32).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}_j(z) = 0\}$
$m = 2$, $n_0 = 2$	$r = 1$, $m_0 = m = 2$ $\beta_0 = 1$	$\mathcal{P}_0(z) := z^2 + \frac{4\boldsymbol{i}\pi}{2\pi^2-8}z - \frac{2}{2\pi^2-8}$
$\Pi = \{(0,2),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,2),(1,1),(2,0)\}$	$\{c_{0,1} = -\frac{\boldsymbol{i}}{\pi+2},\ c_{0,2} - \frac{\boldsymbol{i}}{\pi-2}\}$

After applying the Algorithm 1, we obtain the results summarized in Table 2.3. Since $n_0 > n_1$ we are able to use Proposition 2.3.3, to obtain the first approximation of the Weierstrass polynomial $W(s,\tau) = s^2 + w_1 s + w_0$. Thus, the coefficients $w_i(\tau)$ are given by

$$w_0(\tau) = \frac{-2}{2\pi^2 - 8}\tau^2 + o(\tau^2), \quad w_1(\tau) = \frac{4\boldsymbol{i}\pi}{2\pi^2 - 8}\tau + o(\tau).$$

Now, following Proposition 2.4.2 the solutions of the quasi-polynomial $f(s,\tau)$ (2.32) around the critical pair $(s^*, \tau^*) = (\boldsymbol{i}, \pi)$ have only one branch ($r = 1$). Moreover, since $\beta = 1$ and the solutions of $\mathcal{P}_0$, $c_{0,1}$ and $c_{0,2}$ are simple roots ($\mu = 1$), Proposition 2.4.4 implies that the solutions can be expanded as the following Taylor series:

$$s_{0,1}(\tau) = \boldsymbol{i} - \frac{\boldsymbol{i}}{\pi + 2}(\tau - \pi) + o(\tau - \pi), \quad s_{0,2}(\tau) = \boldsymbol{i} - \frac{\boldsymbol{i}}{\pi - 2}(\tau - \pi) + o(\tau - \pi).$$

Finally, we have that $m_0 = 2$ and $\beta_0 = 1 > 1/m_0$ then the solution $s = \boldsymbol{i}$ posses

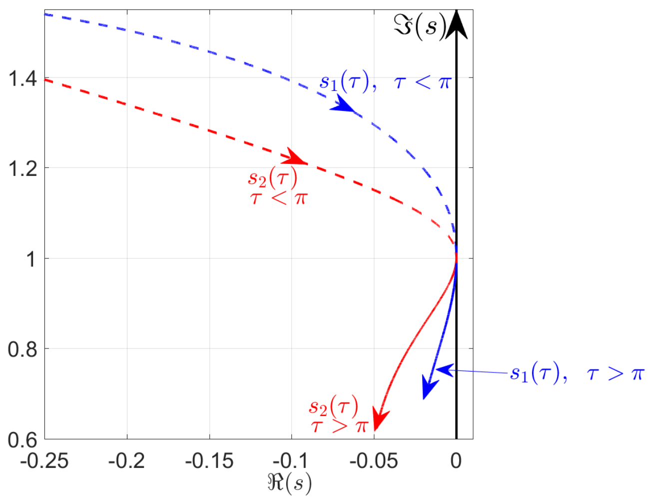


Figure 2.4: Root locus behavior, in a degenerate case, for quasi-polynomial (2.32)

the NRS property. The asymptotic behavior is illustrated in Fig. 2.4.

Example 2.5.3. *Consider the following quasi-polynomial:*

$$f(s,\tau) = \left(s^4 + 3s^2 + 2\right) + \left(s^2 + 1\right) e^{-s\tau}, \tag{2.33}$$

with critical root at $(\boldsymbol{i}, \pi)$ *and multiplicity* $m = 2$. *For this example we have that*

$$\left.\frac{\partial^{n_0} f}{\partial \tau^{n_0}}\right|_{(\boldsymbol{i},\pi)} = 0,\ \forall n_0 \in \mathbb{N} \Rightarrow n_0 = \infty, \qquad \left.\frac{\partial^2 f}{\partial \tau \partial s}\right|_{(\boldsymbol{i},\pi)} = -2 \Rightarrow n_1 = 1.$$

Since n_0 *is not bounded and* n_1 *is finite, we have that* $\kappa = 1$, *implying that we have* $1-$*invariant solution at* $s = \boldsymbol{i}$ *(denoted as* $s_{\kappa,1}$ *in Fig. 2.5). Therefore, under these conditions the Weierstrass polynomial associated to* $f(s,\tau)$ *at* $(\boldsymbol{i}, \tau)$ *is given by*

$$W(s,\tau) = (s - \boldsymbol{i})\left((s - \boldsymbol{i}) + w_1(\tau)\right).$$

For the remaining solution, we apply Algorithm 1. Table 2.4 summarizes the results. Next, in the light of Proposition 2.4.3, we have:

Table 2.4: Results summary for the quasi-polynomial (2.33).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} :\ \mathcal{P}_j(z) = 0\}$
$m = 2,\ n_1 = 1$	$r = 1,\ m_0 = 1,\ \beta_0 = 1$	$\mathcal{P}_0(z) := z - \frac{4}{-8+4\boldsymbol{i}\pi}$
$\Pi = \{(1,1),(2,0)\}$	$\Pi^{(0)} = \{(1,1),(2,0)\}$	$\left\{c_{0,1} = \frac{1}{-2+\boldsymbol{i}\pi}\right\}$

$$s_1(\tau) = \boldsymbol{i} - \frac{2 + \boldsymbol{i}\pi}{4 + \pi^2}(\tau - \pi) + o(\tau - \pi).$$

In addition, since $\beta = 1$ *and* c_0 *is a simple root of* $\mathcal{P}_0$, *by a direct application of Proposition 2.4.4 we conclude that* s_1 *can be expanded as a Taylor series. Such a behavior is illustrated in Figure 2.5.*

Example 2.5.4. *Inverted Pendulum*

The Inverted Pendulum is one of the most important classical problems of Control Engineering. In a wide variety of situations, the need to implement the derivative action arises, which not realizable. An alternative can be to use the time-delay action in the controller. Its behavior using time-delay in the feedback has been studied in the literature, see, for instance, [9]. Thus, the quasi-polynomial of the inverted pendulum using time-delay action is given by

$$f(s,\tau) = \left(s^2 + k\right) - ae^{-s\tau} - be^{-2s\tau}, \tag{2.34}$$

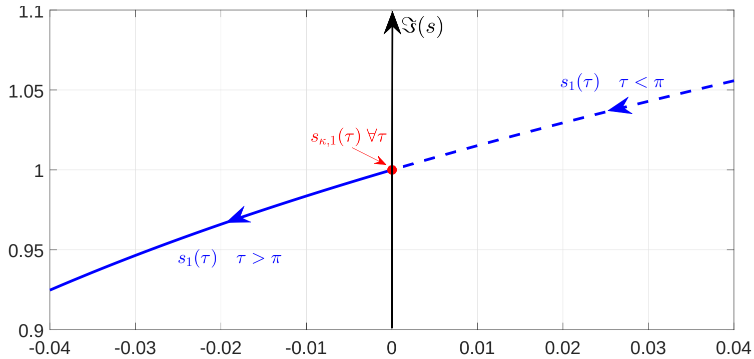


Figure 2.5: Root locus behavior, in a degenerate case, for quasi-polynomial (2.33).

where $k = -g/l$ and a, b are the controller constants. In order to facilitate analysis, the focus is on a multiple zero spectral value, $s = 0$. By setting $k = a + b$ the quasi-polynomial f vanish, moreover for $a = -2b$ the first partial derivative $\partial_s f$ also vanish. Now, considering the delay interval $0 < \tau^2 < 2(a + 4b)$, the root at the origin $s^* = 0$ has multiplicity $m = 2$. In this manner, the corresponding Weierstrass polynomial has the form $W(, \tau) = s^2 + w_1(\tau)s + w_0(\tau)$. For the computation of the Weierstrass polynomial, the first partial derivatives of f at $s^* = 0$ and $\tau^* = 2(a + 4b)^{-1}$ are computed:

$$\left.\frac{\partial^{n_0} f}{\partial \tau^{n_0}}\right|_{s^*=0} = 0, \forall n_0 \in \mathbb{N} \Rightarrow n_0 = \infty \qquad \left.\frac{\partial^{n_1+1} f}{\partial \tau^{n_1} \partial s}\right|_{s^*=0} = 0, \forall n_1 \in \mathbb{N} \Rightarrow n_1 = \infty, \tag{2.35}$$

implying that $\kappa = 2$. In the light of (2.35), the Weierstrass coefficient w_0 and w_1 are identically zero, thus there are 2-invariant solutions $s = 0$. Thus, around the origin the quasi-polynomial (2.34) can be written as

$$f(s, \tau) = s^2 b(s, \tau).$$

Continuing with the choice of parameters $k = a + b$ and $a = -2b$, the delay is set $(\tau^*)^2 = 2(a + 4b)^{-1}$. Such critical delay value corresponds to a multiplicity $m = 3$. Thus, following the Weierstrass Preparation Theorem, the local behavior of the quasi-polynomial is captured by

$$f(s, \tau) = \left[s^3 + w_2(\tau)s^2 + w_1(\tau)s + w_0(\tau)\right] b(s, \tau).$$

As in the same way as in the case of the double root $n_0 = n_1 = \infty$ (2.35)*, which leaves the following derivatives to be determined:*

$$\left.\frac{\partial^3 f}{\partial\tau\partial s^2}\right|_{(s^*,\tau^*)} = -2\tau(a+4b) = -4\sqrt{b}, \Rightarrow n_2 = 1 \qquad \left.\frac{\partial^3 f}{\partial s^3}\right|_{(s^*,\tau^*)} = \frac{6}{\sqrt{b}}. \quad (2.36)$$

Therefore the Weierstrass coefficient w_2 *is non-zero, meaning that we must have* $\kappa = 2$*, that is, in this situation, we have 2-invariant solutions at* $s = 0$ *and a solution* $s(\tau) = -w_2(\tau)$*. The first term in the expansion of coefficient* w_2 *can be obtained by using* (2.36) *and Proposition 2.3.3 resulting in the following Weierstrass polynomial*

$$W(s,\tau) = s^2\left[s + w_2(\tau)\right].$$

Therefore, two roots $s_{1,2} \equiv 0$ *remain invariant under delay variations, and the third root is given by* $s_3(\tau) = -w_2(\tau)$*. With this results, the Newton polygon has a single segment with two points as shown in Table 2.5. Now, the asymptotic*

Table 2.5: Results summary for the quasi-polynomial (2.34).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} :\ \mathcal{P}_j(z) = 0\}$
$m = 3$, $n_2 = 1$	$r = 1$, $m_0 = 1$, $\beta_0 = 1$	$\mathcal{P}_0(z) := z - 2b$
$\Pi = \{(2,1),(3,0)\}$	$\Pi^{(0)} = \{(2,1),(3,0)\}$	$\{c_{0,1} = 2b\}$

behavior of the of the solution s_3 *is obtain by means of Proposition 2.4.3 Table 2.5, resulting in*

$$s_3(\tau) = 2b\tau + o(\tau).$$

as depicted in figure 2.6.

Example 2.5.5. *Consider the quasi-polynomial:*

$$f(s,\tau) = \left(s^6 + 3s^4 + 3s^2 + 2\right) + 2e^{-s\tau} + e^{-2s\tau}, \quad (2.37)$$

where $s = \boldsymbol{i}$ *is a double root at* $\tau = \pi$*. As in the previous examples, we have:*

$$\left.\frac{\partial^2 f}{\partial\tau^2}\right|_{(\boldsymbol{i},\pi)} = -2 \Rightarrow n_0 = 2, \qquad \left.\frac{\partial^2 f}{\partial\tau\partial s}\right|_{(\boldsymbol{i},\pi)} = 2\boldsymbol{i}\pi \Rightarrow n_1 = 1.$$

After applying Algorithm 1 along with Proposition 2.4.3, Table 2.6 summarizes the results.

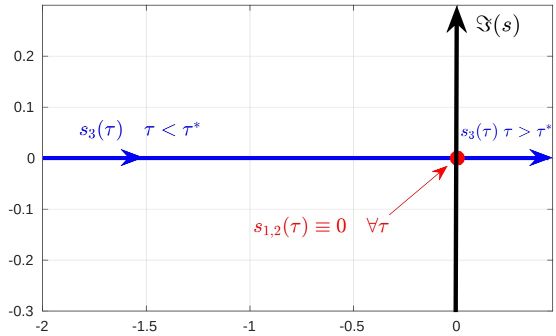


Figure 2.6: Root locus behavior, with a 2-invariant roots, for quasi-polynomial (2.34) with $k = -0.5$, $a = -1$ and $b = 0.5$.

Table 2.6: Results summary for the quasi-polynomial (2.37).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}(z) = 0\}$
$m = 2$, $n_0 = 2$	$r = 1$, $m_0 = 2$, $\beta_0 = 1$, $i_{-1} = 0$	$\mathcal{P}_0(z) := z^2 + \frac{2\boldsymbol{i}}{\pi} z - \frac{1}{\pi^2}$
$\Pi = \{(0,2),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,2),(1,1),(2,0)\}$	$\left\{c_{0,1} = \frac{-\boldsymbol{i}}{\pi}\right\}$

Observe that since $\Re(c_{0,1}) \equiv 0$, we do not have enough information to determine the crossing directions. Now, from Table 2.6, it can be seen that $\beta_0 = 1$ and $c_{0,1}$ is a multiple solution of $\mathcal{P}_0$ with multiplicity $m_0 = 2$. Hence, in the light of Proposition 2.4.6, we compute $\frac{d^{\nu_j+1}}{\tau^{\nu_j+1}} f\left(c_{j\sigma}\tau, \tau\right)$. Taking into account that $\nu_0 = n_{i_{-1}} + i_{-1} = 2$, we have:

$$\frac{d^3}{d\tau^3} f\left(-\frac{\boldsymbol{i}\tau}{\pi}, \tau\right)\bigg|_{\tau=0} = \frac{48}{\pi^3}.$$

Since we fulfill all hypothesis of Proposition 2.4.6, we are able to compute the higher-order terms. For this purpose, let us consider the function f_1 (see Remark 2.4.4):

$$f_1\left(s_1, \tau\right) := \frac{1}{\tau^2} f\left(\tau\left(s_1 - \frac{\boldsymbol{i}}{\pi}\right), \tau\right), \tag{2.38}$$

where it is assumed that the critical point $(\boldsymbol{i}, \pi)$ has been shifted to the origin. Thus, a direct application of Algorithm 1 to f_1 gives the results summarized in Table 2.7. According to Proposition 2.4.6, the solutions of (2.37) can be expanded

Table 2.7: Results summary for the quasi-polynomial (2.38).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}(z) = 0\}$
$m = 2,\ n_0 = 1$	$r = 1,\ m_0 = 2,\ \beta_0^{(1)} = 1/2$	$\mathcal{P}_0(z) := z^2 + \frac{8}{\pi^5}$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\left\{c_{0,\sigma} = -(-1)^\sigma \boldsymbol{i} \left(\frac{8}{\pi^5}\right)^{1/2}\right\}$

as:

$$s_\sigma(\tau) = \boldsymbol{i} - \frac{\boldsymbol{i}}{\pi}(\tau - \pi) + (-1)^\sigma \boldsymbol{i}\sqrt{\frac{8}{\pi^5}}(\tau - \pi)^{1+1/2} + o\left((\tau - \pi)^2\right), \quad \sigma = 1, 2.$$

Finally, since $\beta_0 = 1 > 1/m_0$ the solution $s = \boldsymbol{i}$ has the NRS property, this behavior is illustrated in Fig. 2.7.

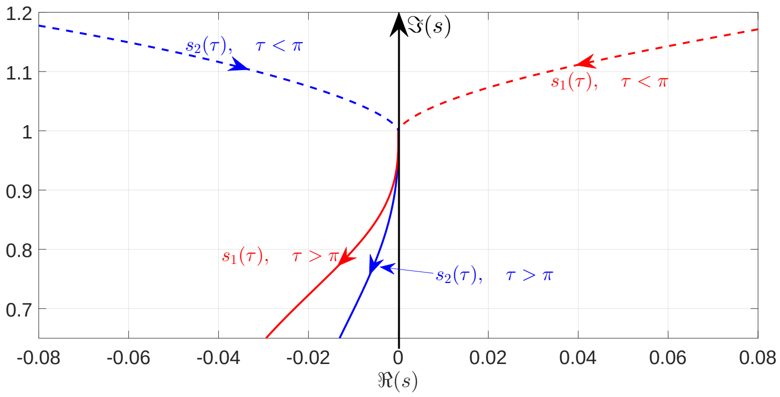


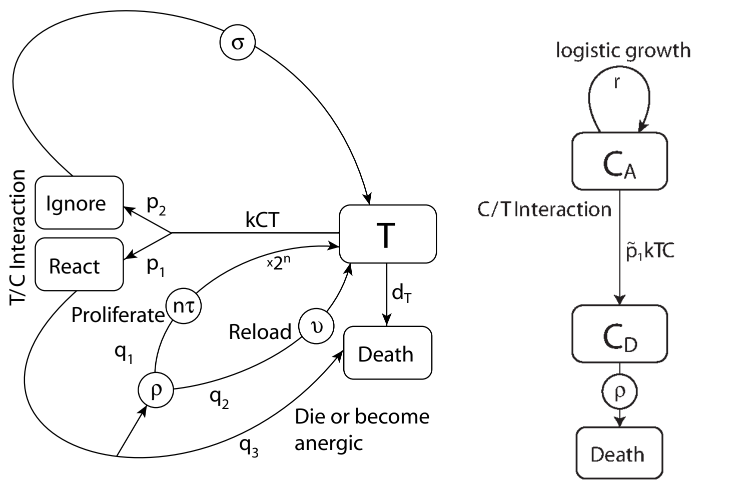
Figure 2.7: Double root with NRS property of the quasi-polynomial (2.37). Since the leading term is purely imaginary, what keeps one root tangent to the imaginary axis, thus higher-order terms are computed for the splitting of these two roots.

One of the most important aspects of time-delay systems is that they allow capturing accurately the behavior of several complex systems arising in nature. For such a reason, in the remaining part of this chapter, we will consider some biological systems that include delays in their model. In this vein, we will illustrate the relevance of the preceding results when analyzing some stability issues that can emerge in these systems.

Example 2.5.6. *Immune Response to Chronic Myelogenous Leukemia ([74])*

Consider a model for the post-transplantation dynamics of the immune response to chronic Myelogenous Leukemia. Such a model reflects the evolution and the interaction between the anti-cancer cells T and the total population of cancer cells C. In order to consider a more realistic situation, the authors decompose the total population as active population C_A and dying population C_D, that is, $C(t) = C_A(t) + C_D(t)$. Figure 2.8 schematically illustrated the evolution of T cells and C cells.

To analyze the behavior around its equilibrium points, the authors consider the



(a) Evolution of T cells (b) Evolution of C cells

Figure 2.8: Stage evolution of anti-cancer cells and cancer cells.

following linearized system:

$$\begin{cases} \dfrac{dT(t)}{dt} = -b_1 T(t) - b_2 T_0 \left(C_A(t) + C_D(t)\right) \\ \qquad + b_3 \left(C_{A,0} + C_{D,0}\right) T(t-\sigma) + b_3 T_0 \left(C_A(t-\sigma) + C_D(t-\sigma)\right) \\ \qquad + b_4 \left(C_{A,0} + C_{D,0})\right) T(t-\tau) + b_4 T_0 \left(C_A(t-\tau) + C_D(t-\bar{\tau})\right) \\ \qquad + b_5 \left(C_{A,0} + C_{D,0})\right) T(t-\bar{v}) + b_5 T_0 \left(C_A(t-\bar{v}) + C_D(t-v)\right), \\ \dfrac{dC_A}{dt} = c_1 C_A(t) - c_3 C_{A,0} T(t) - c_3 T_0 C_A(t), \\ \dfrac{dC_D}{dt} = c_3 C_{A,0} T(t) + c_3 T_0 C_A(t) - c_3 C_{A,0} T(t-\rho) - c_3 T_0 C_A(t-\rho). \end{cases}$$

Hence, the stability analysis can be performed by studying the quasi-polynomial given by:

$$\tilde{f}(s;\boldsymbol{\tau}) = p_0(s,\rho) + p_1(s,\rho)e^{-s\sigma} + p_2(s,\rho)e^{-s\tau} + p_3(s,\rho)e^{-sv},$$

where the coefficients are expressed as:

$$\begin{aligned}
p_0(s,\rho) &:= -s^3 + (-b_1 + c_1 - c_3T_0)s^2 + (b_1c_1 - b_1c_3T_0 - b_2c_3C_{D,0}T_0)s \\
&\quad - (b_2c_3C_{A,0}T_0)s(1-e^{-s\rho}) + (b_2c_1c_3C_{A,0}T_0)(1-e^{-s\rho}), \\
p_k(s,\rho) &:= b_{(k+2)}C_Ap_{aux}(\rho,s), \quad k = 1,2,3, \\
p_{aux}(s,\rho) &:= (1+\rho c_3)s^2 + (\rho c_3T_0 - c_1(1+\rho c_3))s + (c_3T_0)s(1-e^{-s\rho}) - \\
&\quad - (c_1c_3T_0)(1-e^{-s\rho}).
\end{aligned}$$

In the previous definition T_0 denotes the initial cell population; $C_{A,0}$, $C_{D,0}$ the active and dying population, respectively. The parameters $b_i, c_i \in \mathbb{R}$, $i \in \{1,2,3\}$, are related with the kinetic constant of the interaction rate, the probability of reaction and proliferation and the net growth rate (see [74] for further details).
Since the interest is on critical roots $\boldsymbol{i}\omega^*$, the complexity of the problem can be reduced if the following condition holds

$$|p_2(\boldsymbol{i}\omega^*,\rho)| = |p_3(\boldsymbol{i}\omega^*,\rho)|,$$

in such a way that the quasi-polynomial is reduced. Now, setting $\rho = 0$, we obtain a quasi-polynomial with a single delay:

$$\begin{aligned}
f(s,\sigma) = s^3 &+ (b_1 - c_1 + c_3T_0)s^2 + (b_1c_3T_0 - b_1c_1 + b_2c_3C_{D,0}T_0)s - \\
&b_3C_{A,0}(s^2 - c_1s)e^{-s\sigma}.
\end{aligned} \tag{2.39}$$

Now, choosing the parameters such that $c_1 = 0$, $\pi(b_1 + c_3T_0) = -\sqrt{2}$, $b_1c_3T_0 + b_2c_3C_{D,0}T_0 = 2$ and $-\pi b_3C_{A,0} = \sqrt{2}$, the quasi-polynomial f has a critical solution at $s^* = \boldsymbol{i}\sqrt{2}$ with multiplicity $m = 2$ at $\sigma = \sqrt{2}\pi$. To start the method, the first non-zero partial derivatives are obtained:

$$\begin{aligned}
\left.\frac{\partial f}{\partial \sigma}\right|_{(\boldsymbol{i}\sqrt{2},\sqrt{2}\pi)} &= \boldsymbol{i}\frac{4}{\pi}, \qquad \left.\frac{\partial^2 f}{\partial s \partial \sigma}\right|_{(\boldsymbol{i}\sqrt{2},\sqrt{2}\pi)} = \frac{6\sqrt{2}}{\pi} - \boldsymbol{i}4\sqrt{2}, \\
\left.\frac{\partial^2 f}{\partial s^2}\right|_{(\boldsymbol{i}\sqrt{2},\sqrt{2}\pi)} &= -2\sqrt{2}(2\pi + \boldsymbol{i}),
\end{aligned}$$

hence, the input of the algorithm is $n_0 = 1$, n_1 *and* $m = 2$. *Therefore, following similar steps than those presented in the previous examples, Table 2.8 summarizes the results obtained by applying the Algorithm 1 in conjunction with Proposition 2.4.1 and Proposition 2.4.3.*

Now, the Puiseux series solutions According to Proposition 2.4.6, the solutions of

Table 2.8: Results summary for the quasi-polynomial (2.39).

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}_j(z) = 0\}$
$m = 2$, $n_0 = 1$	$r = 1$, $m_0 = m = 2$ $\beta_0 = 1/2$	$\mathcal{P}_0(z) := z^2 - \frac{2\boldsymbol{i}\sqrt{2}}{\pi(2\pi+\boldsymbol{i})}$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\left\{c_{1,2} \approx \pm\frac{(1+i)\sqrt[4]{2}}{\sqrt{\pi(2\pi+i)}}\right\}$

(2.39) *can be expanded as*

$$s_k(\sigma) = \boldsymbol{i}\sqrt{2} + (-1)^k \left(\frac{(1+i)\sqrt[4]{2}}{\sqrt{\pi(2\pi+i)}}\right)(\sigma - \sqrt{2}\pi)^{1/2} + o\left(|\sigma - \sqrt{2}\pi|^{1/2}\right), \quad k = 1, 2,$$

as shown in Figure 2.9.

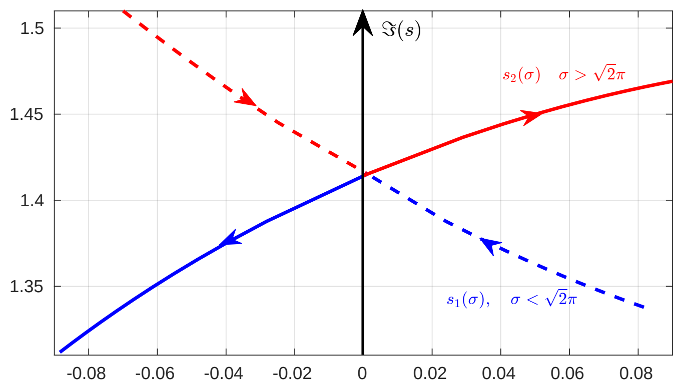


Figure 2.9: CRS of critical $(\boldsymbol{i}\pi, 2)$ of quasi-polynomial (2.39).

Example 2.5.7. *Competition in a Chain of Chemostats ([62])*

As final example of this chapter, let's borrowed a model of competition in a chain of chemostats with delay. The system proposed by the authors considers two chemostats, one contains two microbial species in competition for a single *limiting nutrient and receives an external input of the less advantaged competitor, which is cultivated in an external chemostat. The coupled system, when the output of the first chemostat becomes the input the second containing two competitors, describes the evolution of the concentration of the* i*th species in the* j*th chemostat competing for the* i*th nutrient in the* i*th chemostat. Now, the coupled system for two microbial species with densities* x_{ij} *competing for a single limiting substrate with concentration* s *determine the system:*

$$\begin{aligned}
\dot{s}_1(t) &= D[s^0 - s_1(t)] - \mu_1(s_1(t))x_{11}(t)\\
\dot{x}_{11}(t) &= x_{11}(t)\mu_1(s_1(t-\tau)) - Dx_{11}(t)\\
\dot{s}_2(t) &= D[s_1(t) - s_2(t)] - \mu_1(s_2(t))x_{12}(t) - \mu_2(s_2(t))x_{22}(t)\\
\dot{x}_{12}(t) &= x_{12}\mu_1(s_1(t-\tau)) - D[x_{11}(t) - x_{12}(t)]\\
\dot{x}_{22}(t) &= x_{22}\mu_2(s_2(t-\tau)) - Dx_{22}(t)
\end{aligned}$$

The local stability is performed by means of the associated quasi-polynomial:

$$\begin{aligned}
f(s,\tau) = &\left[s^5 + (a+\mathcal{A})s^4 + (b+a\mathcal{A})s^3 + (b\mathcal{A})s^2\right] +\\
&\left[(\mathcal{B}+c)s^3 + (a\mathcal{B} + \mathcal{A}c + d)s^2 + (\mathcal{A}d + b\mathcal{B})s\right]e^{-s\tau} +\\
&\left[(\mathcal{B}c)s + \mathcal{B}d\right]e^{-2s\tau},
\end{aligned} \tag{2.40}$$

where the positive constants $\mathcal{A}$*,* $\mathcal{B}$*,* a*,* b*,* c *and* d *depict the dilution rate and the per-capita growth and consumption of nutrient. In order to illustrate the effectiveness of the proposed methodology, we consider the following set of parameters* $a = -1$*,* $b = (4+\pi^2)/4$*,* $c = 1$*,* $d = \pi^2/4$ *and* $\sqrt{2}\,(-\mathcal{A}^2 + (\mathcal{A}^4 + 4\mathcal{B}^2))^{1/2} \neq \pi$*. For* $\tau^* = 2$ *there exists a double critical solution at* $s^* = \boldsymbol{i}\pi/2$*, such that the quasi-polynomial* (2.40) *satisfy the following partial derivatives:*

$$\left.\frac{\partial f}{\partial \tau}\right|_{\left(\boldsymbol{i}\frac{\pi}{2},2\right)} = -\frac{\pi^2}{4} + \boldsymbol{i}\frac{\pi^3}{8}, \qquad \left.\frac{\partial^2 f}{\partial s \partial \tau}\right|_{\left(\boldsymbol{i}\frac{\pi}{2},2\right)} = \frac{3}{4}\pi + \boldsymbol{i}\frac{4\pi - \pi^3}{4},$$
$$\left.\frac{\partial^2 f}{\partial s^2}\right|_{\left(\boldsymbol{i}\frac{\pi}{2},2\right)} = 2 - \pi^2 + \boldsymbol{i}\pi,$$

which results in $n_0 = 1$*,* n_1 *and* $m = 2$*. These computations are sufficient to obtain the first term in the expansion. Applying the Algorithm 1, the exponent* β_0 *is determined, the coefficients are computed using Proposition 2.4.1 and the splitting* *properties are given following Proposition 2.4.3. The results are summarized in Table 2.9. According to Proposition 2.4.6, the solutions can be expanded as*

Table 2.9: Results summary for the quasi-polynomial of the Chemostat.

Initial Data	Algorithm Output	$\mathcal{Z} := \{z \in \mathbb{C} : \mathcal{P}(z) = 0\}$
$m = 2,\ n_0 = 1$	$r = 1,\ m_0 = 2,\ \beta_0 = 1/2$	$\mathcal{P}_0(z) := z^2 - 0.88 + \boldsymbol{i}0.6337$
$\Pi = \{(0,1),(1,1),(2,0)\}$	$\Pi^{(0)} = \{(0,1),(2,0)\}$	$\{c_{0,1} \approx \pm(0.3197 + \boldsymbol{i}0.9911)\}$

$$s_\sigma(\tau) = \boldsymbol{i}\frac{\pi}{2} + (-1)^\sigma(0.3197 + \boldsymbol{i}0.9911)(\tau - 2)^{1/2} + o\left(|\tau - 2|^{1/2}\right), \quad \sigma = 1, 2.$$

following Proposition 2.4.2, CRS property is present in the splitting behavior as shown in 2.10.

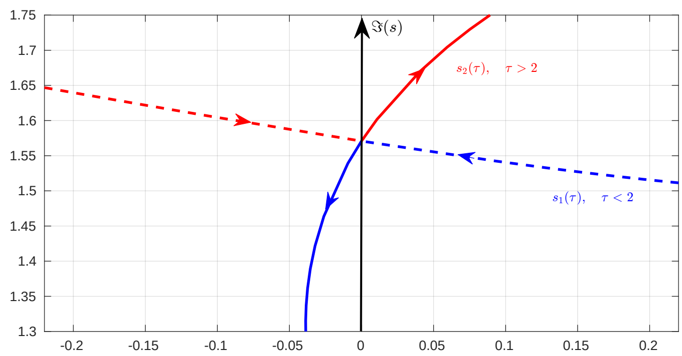


Figure 2.10: Splitting of double root with CRS property of quasi-polynomial $f(s,\tau)$ (2.40) around $(\boldsymbol{i}\pi/2, 2)$.

2.6 Chapter Summary

This chapter is devoted to the analysis of the asymptotic behavior of multiple imaginary roots for quasi-polynomials of retarded-type. We pursue to extend the study already carried out in the literature, which motivates us to the analysis of multiple roots, although not common, multiple roots are necessary to take into account in order to understand all possible dynamical behaviors of dynamical systems.

Compared to published literature on this particular topic, the work here present contributes deeply explores the intrinsic relation between the series expansion and the Weierstrass polynomial. This in order to derive both, the Weierstrass polynomial and series expansion without additional steps. Furthermore, several examples are given to show the applicability of the proposed algorithm. Moreover, it is worth mentioning that the approach proposed here has as its starting point the quasi-polynomial of the system. The effective computation of simple and multiple critical roots is a rich and important numerical problem, even in the polynomial case.

Some algebraic properties have been presented to characterize the branch structure for all critical solutions of the quasi-polynomial $f(s,\tau)$. In addition, we have shown that the leading terms of the Puiseux (or Taylor) expansions can be derived in a simple manner by computing the solutions of a given polynomial $\mathcal{P}_j$. We have also presented a characterization for the critical solutions (CRS, RS, NRS). Such classification has been shown to be extremely useful in analyzing the stability behavior of such solutions. Insights concerning the higher-order terms for the Puiseux (or Taylor) expansion of the critical solutions are also given. Such characterizations have been shown to be very convenient to determine the series expansion nature.

Chapter 3

Extension to Two Delay Parameters

Multiple delays can also be presented in a more general form, as non-commensurate delays, that is, all delays are assuming to be independent of each other. In this case, the stability analysis of the related quasi-polynomial becomes a more complex task. This means that the problem is less studied for technical reasons. For instance, the case of solutions of multiplicity two is considered in [41], where the authors have proposed two sectors to analyze the qualitative properties of these roots when delays are subject to small deviations and restricted to such sectors. Chapter 2 deals with singularities of plane curves, in particular, multiple roots of the equation $f(z, \tau) = 0$, nevertheless, singularities of higher dimension do not always possess a parametrization. In this vein, it is worth mentioning that unlike the case of a single parameter, in the multi-parameter case there exist some *singular* and *unexpected* behaviors (see, the motivating example) which have to be taken into account (see, for instance, [67]), in order that the problem is well-posed. In other words, the Puiseux type arguments cannot be extended straightforwardly from one parameter to a multi-parameter case. As in the previous chapter the critical root $(s^*, \boldsymbol{\tau}^*)$ is translated translated to the origin by the shifts $s \mapsto s - s^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$, $\tau_1 \mapsto \tau_1 - \tau_1^*$.

3.1 Some Motivating Examples

Even though we can reduce the analysis of a given entire function f to the study of an algebraic function W (the associated Weierstrass polynomial), in this section we aim to point out some difficulties that arise regarding multi-parameter functions. In order to illustrate such arguments, let us give the following motivating example.

Example 3.1.1. *Consider the following perturbed polynomial:*

$$P(z, x_1, x_2) = z^2 + 3x_1 z + 2\left(x_1^2 + 2x_2^2\right), \tag{3.1}$$

where x_1 and x_2 are considered as perturbation parameters. It is clear to see, that for $x_1 = x_2 = 0$, $z = 0$ is a root of multiplicity two. In this case, the solutions $z_{1,2}(\mathbf{x})$ are not analytic at $\mathbf{x} = (0,0)$. Furthermore, $z_k(\mathbf{x})$ does not have a unique *representation as a power series which is convergent in some a particular neighborhood of the origin (the appropriate regions are shown in figure 3.1). In order to illustrate this assertion, let us consider the region $|x_1| < |x_2|$, in this region the solutions for $k \in \{1,2\}$ admit the following representation*

$$\begin{aligned} z_k(\mathbf{x}) \;=\; & -\frac{1}{2}\left(3x_1 + (-1)^k \boldsymbol{i} 4x_2\right) + \frac{(-1)^k}{16} x_1 \left(\boldsymbol{i}\frac{x_1}{x_2} + \frac{\boldsymbol{i}}{64}\left(\frac{x_1}{x_2}\right)^3 + \right. \\ & \left. \frac{\boldsymbol{i}}{2048}\left(\frac{x_2}{x_1}\right)^5 + \mathcal{O}\left(\left(\frac{x_2}{x_1}\right)^5\right)\right). \end{aligned}$$

Now, if instead of the previous region, we consider the region $|x_2| < |x_1|$, then the solutions admit the following representation

$$z_k(\mathbf{x}) = -2^{k-1}x_1 + (-1)^k 4x_2\left(\frac{x_2}{x_1} + 4\left(\frac{x_2}{x_1}\right)^3 + 32\left(\frac{x_2}{x_1}\right)^5 + \mathcal{O}\left(\left(\frac{x_1}{x_2}\right)^5\right)\right).$$

The arguments in the example above clearly have shown that some further considerations must be taken into account in the expansion of the roots as generalized fractional power series (see, Section 1.5).

As discussed in the previous chapter, the Newton diagram is a powerful tool to analyze the asymptotic behavior of multiple roots. However, in order to be able to apply such a procedure to the multi-parameter case, some special situations

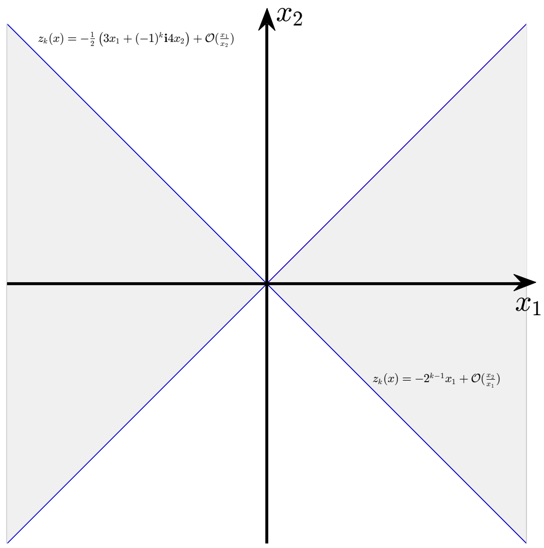


Figure 3.1: Regions of convergence of solution $z_k(\mathbf{x})$. The white region describes the cone $|x_1| < |x_2|$ and gray region the cone $|x_2| < |x_1|$.

must be taken into consideration, as discussed in [5]. In essence, two types of segments in Newton's polygon and two types of solutions series may be arise when generalizing the method, as shown in the following example.

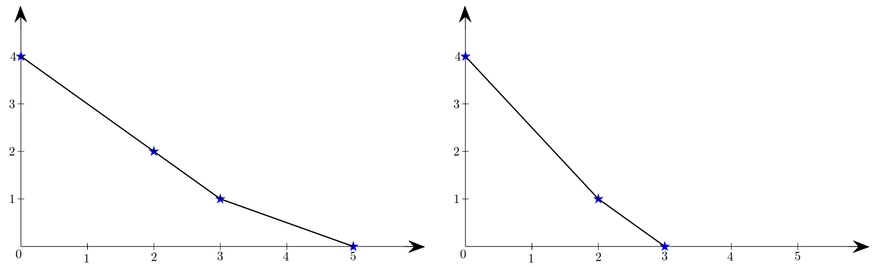
Example 3.1.2. *The equation $P(z, \mathbf{x}) = 0$ posses solutions with multiplicity greater than one, and it is determined by the perturbed polynomial:*

$$P(z, \mathbf{x}) := z^5 + \left(x_1 x_2^3 + x_1^2 x_2^2\right) z^3 + \left(x_1^2 x_2^2 + x_1^3 x_2\right) z^2 + \left(x_1^4 x_2\right). \qquad (3.2)$$

Clearly, $z = 0$ is a $5-$multiple root at $\mathbf{x} = (0, 0)$. Now, let us form the Newton diagram with respect to x_1, obtaining $\Pi = \{(0, 4), (2, 2), (3, 1), (5, 0)\}$, illustrated in figure 3.2-(a). The slope $\beta_0 = 1$ determines $3-$solutions with respect to x_1, and coefficients that are solutions of the polynomial:

$$\mathcal{P}(\xi) = x_2 + x_2^2 \xi^2 + x_2^3 \xi^3 = 0.$$

In this case, it is clear that the solutions cannot be easily computed. In order to compute solutions by applying the Newton procedure, a monic polynomial is



(a) Newton polygon with respect to x_1.(b) Newton polygon with respect to y_2.

Figure 3.2: Newton polygons for $P(z, \boldsymbol{\epsilon})$ (3.2).

considered. Thus, with the aim of overcoming such difficulty, let us consider the change of variables (blowing-ups):

$$\zeta := \xi \quad x_1 = v_1 x_2 \quad x_2 = v_2,$$

and

$$v_1 = y_1 v_2 \quad v_2 = y_2.$$

These changes of variables enable us to avoid horizontal segments in the subsequent steps of the process. The resulting polynomial $P'(y_1, y_2)$ posses the same Newton polygon, and the segment with $\beta_0 = 1$ has a monic polynomial

$$y_2^{-5}\mathcal{P}'(\xi, y_2) = y_2^4 + y_2\xi^2 + \xi^3 = 0.$$

Applying Newton procedure, (see figure 3.2-(b)) to the above equation, we derive the fractional power series solution of P':

$$\zeta_1(y_1, y_2) = -y_2 y_1 + o(y_1 y_2),$$
$$\zeta_{2,3}(y_1, y_2) = \pm \boldsymbol{i} y_2^{2/3} y_1 + o\left(y_1 y_2^{1/3}\right).$$

From the examples above, the need to properly extend the methodology proposed in Chapter 2 is shown. On the one hand the appropriate construction of the solutions series using the Newton method. On the other hand, the correct definition and description of the generalization of the Puiseux series.

3.2 Construction of Generalized Puiseux Series Solutions

The following notation and terminology will allow defining fractional power series solutions known as the Puiseux series. We must point out that in the case of multi-parameter algebraic equations $f(z, \mathbf{x}) = 0$ we must be rigorous in defining them and describing their structure (these results were published in [59]). Let us be more specific, we deal with singularities of greater dimension in particular with multiple roots of the polynomial P as a function of the parameters $\mathbf{x}$. We must point out that the power series proposed as a solution must satisfy certain conditions, we must use a ring of multi-variable fractional power series. In [63] the author defines a fractional power series ring that contains the solutions of algebraic hyper-surfaces. This can be achieved through formal power series defined in a geometric way, by taking infinite power series:

$$\sum_{i=1}^{\infty} c_{\boldsymbol{a_i}} \boldsymbol{x}^{\boldsymbol{a}_i/d}, \quad \text{where } \boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n},$$

where the exponents $\boldsymbol{a}$ are taken from a fixed convex cones Definition 1.5.1 in Chapter 1 with structure related to its Newton polytopes [15], so that they are in the form of *Generalized Puiseux Power Series* Section 1.5.

3.2.1 Computation of Weierstrass Polynomial

In [55], the authors propose a method to compute the Weierstrass polynomial for an holomorphic function. This method is based on its partial derivatives and combinatorial factors related in a recursive way. For the case of holomorphic function, $f(z, \boldsymbol{x})$ of complex variables with $\boldsymbol{x} = (x_1, x_2)$ and $z = 0$ a m-multiple root at $(x_1, x_2) = (0, 0)$ the computation is given as follows. The coefficients w_i of the Weierstrass polynomials are analytic, $w_i(0, 0) = 0$ and can expressed as convergent power series:

$$w_i(x_1, x_2) = \sum_{h_1+h_2=1}^{\infty} \frac{1}{h_1!h_2!} w_{i,\boldsymbol{h}} x_1^{h_1} x_2^{h_2},$$

where $\boldsymbol{h} = (h_1, h_2)$. It is not difficult to see that the coefficients $w_{i,\boldsymbol{h}}$ can be compute by means of the following partial derivatives:

$$w_{i,\boldsymbol{h}} = \left.\frac{\partial^{h_1+h_2} w_i}{\partial x_2^{h_1} \partial x_1^{h_2}}\right|_{(0,0)}.$$

Since the analytic function locally satisfy $f = Wb$, thus its partial derivatives satisfy the following recursive relations

$$w_{i,\boldsymbol{h}} = \sum_{j=0}^{i} \alpha_{ij} F_{j,\boldsymbol{h}}, \tag{3.3}$$

$$F_{j,\boldsymbol{h}} = f_{j,\boldsymbol{h}} - \sum_{k=0}^{j} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\,(j,k;\boldsymbol{h}',\boldsymbol{h}'')\, w_{k,\boldsymbol{h}'} b_{j-k,\boldsymbol{h}''},$$

with $\boldsymbol{h}' \neq \mathbf{0}$, $\boldsymbol{h}'' \neq \mathbf{0}$ and constant coefficients:

$$\alpha_{jj} = \frac{m!}{j! f_{m,\mathbf{0}}}, \quad \alpha_{ij} = -\frac{m!}{f_{m,\mathbf{0}}} \sum_{k=j}^{i-1} \frac{f_{m+i-k,\mathbf{0}} \alpha_{kj}}{(m+i-k)!},$$

$$c\,(j,k;\boldsymbol{h}_1,\boldsymbol{h}_2) = \frac{j!}{(j-k)!} \prod_{s=1}^{2} \frac{(h'_s + h''_s)!}{h'_s! h''_s!},$$

and for $\boldsymbol{h}' \neq 0$, $k' = k + m$, $b_{k,\boldsymbol{h}}$ is given by

$$\frac{k!}{(m+k)!}\left[f_{k',\boldsymbol{h}} - \sum_{j=0}^{m-1} \sum_{\boldsymbol{h}'+\boldsymbol{h}''=\boldsymbol{h}} c\,(k',j;\boldsymbol{h}',\boldsymbol{h}'')\, w_{j,\boldsymbol{h}'} b_{k'-j,\boldsymbol{h}''}\right].$$

Since we are only interested in the leading terms of w_i, namely a first approximation of the Weierstrass polynomial, we adopt the following notation.

Notation 3.2.1. *Let the natural numbers $n_i^{(j)}$, for $i \in \{0, 1, \cdots, m-1\}$ and $j = 1, 2$, denote the first non zero partial derivative in (z, x_1, x_2) of f, such that the following conditions hold*

$$f(0,0,0) = \frac{\partial^i f}{\partial z^i} = \cdots = \frac{\partial^{i+n_i^{(j)}-1} f}{\partial z^i \partial \tau_j^{n_i^{(j)}-1}} = 0, \quad \frac{\partial^{i+n_i^{(j)}} f}{\partial z^i \partial \tau_j^{n_i^{(j)}}} \neq 0,$$

with derivatives evaluated at $(0,0,0)$. For $n_i^{(j)} = \infty$ we have derivatives:

$$f_{i,(0,0)} = \cdots = f_{i,(0,n'_i-1)} = 0, \quad f_{i,(0,n'_i)} \neq 0,$$

evaluated at $(z, \boldsymbol{x}) = (0, 0, 1)$, with $n'_i \in \mathbb{Z}_+$.

Leading terms of coefficients w_i can be easily found up to the $n_i^{(j)}$ and n'_i derivatives, as a first observation we give the following result.

Proposition 3.2.1. *Suppose that the Weierstrass polynomial has the first non-zero partial derivative, such that*

$$n_i^{(j)} > n_{i+1}^{(j)}, \quad 0 \leq i < m \text{ and } j = 1, 2.$$

Then, the leading terms of $w_i(\boldsymbol{x})$ are given by

$$w_i(x_1, x_2) = \alpha_{i,i} f_{i,(n_i^{(1)},0)} x_1^{n_i^{(1)}} + \alpha_{i,i} f_{i,(0,n_i^{(2)})} x_2^{n_i^{(2)}} + \cdots .$$

If $n_i^{(j)} = \infty$ we get

$$w_i(x_1, x_2) = \alpha_{i,i} f_{i,(n'_i,\eta)} x_1^{n'_i} x_2^{\eta} + \cdots .$$

Proof. This comes directly from equation (3.3), summation over k goes to zero since $w_k, (h_1, 0) = 0$ for $h_1 < n_i^{(1)}$ and $w_k, (0, h_2) = 0$ for $h_2 < n_i^{(2)}$. When $n_i^{(j)} = \infty$, terms $w_k, (h_1, 0) = 0$ and $w_k, (0, h_2) = 0$ for $h_1, h_2 \in \mathbb{N}$. □

As in Section 2.3.3, κ-invariant roots at $z = 0$ for all § arise if

$$f_{i,(h_1,h_2)}\big|_{(0,0,0)} = 0 \quad \forall\, h_1, h_2 \in \mathbb{N},$$

for $i \leq i \leq \kappa - 1$ holds. Thus, according to Theorem 1.3.2 f has the following local structure:

$$z^{\kappa} \left[z^{m-\kappa} + w_{m-\kappa}(\boldsymbol{x}) z^{m-\kappa-1} + \cdots + w_{\kappa}(\boldsymbol{x}) \right] b(z, \boldsymbol{x}).$$

3.2.2 The Newton Diagram Method for Two Parameters

We now consider the monic pseudo-polynomial $f(z, \boldsymbol{x})$ of the form:

$$z^m + a_{m-1}(x_1, x_2) z^{m-1} + \cdots + a_0(x_1, x_2), \tag{3.4}$$

with $a_i(\boldsymbol{x}) \in \mathbb{C}\left[[\boldsymbol{x}]\right]$, such that $f(0, \mathbf{0}) = 0$. The equation $f = 0$ can be solved by applying the Newton diagram method, this is done taking into account just one variable, say x_1, and proceeding iteratively. We take the point π_k as the order of

a_k in x_1, taking x_2 as an element of $\mathbb{C}((x_2))$. For such a purpose, the following definition will be useful

$$\rho_k := \operatorname{ord}_{x_1}(a_k(x_1, x_2)) = \operatorname{ord}(a_k(x_1, 1)). \tag{3.5}$$

Then, the *Newton Polygon* of $f(z, \boldsymbol{x})$, with respect to x_1, is defined by the lower boundary of the convex hull of the points $(k, \rho_k) \in \Pi$ (see, Section 1.4). In order to apply the the Newton diagram procedure, according to Section 1.5, the solution z will take the following structure

$$z(x_1, x_2) = \sum_{i=0}^{\infty} c_i(x_2) x_1^{i/d},$$

where the coefficient $c_i(x_2)$, is in general, given by an univariate Puiseux series in x_2.

3.2.2.1 First Step into the Newton Procedure

Let us suppose that we have determined the Newton diagram of the Weierstrass polynomial (3.4) of f. Since we are dealing with a monic polynomial, the Newton polygon has a finite number of segments, each one with a corresponding set of points $\Pi^{(\ell)}$ and rational numbers $\beta_\ell \geq 0$ satisfying:

$$\beta_0 > \beta_1 > \cdots > \beta_r.$$

Therefore, the segments are presented in two possible ways. The first one, a Newton polygon with a horizontal segment determined by $\beta_r = 0$ and the second one, where $\beta_r > 0$. In this vein, for $0 \leq \kappa < m$, the Newton Diagram Π is given as the set $\Pi = \Pi' \bigcup \Pi''$:

$$\{(0, \rho_0), \ldots, (\kappa, 0)\} \bigcup \{(\kappa, 0), \ldots, (\ell, \rho_\ell), \ldots, (m, 0)\}.$$

Let's take at the first step of the process a horizontal segment with slope $\beta_r = 0$.

Proposition 3.2.2. *Let $f(z, \boldsymbol{x})$ be a pseudo-polynomial with the same structure as* (3.4). *Suppose that at least one coefficient $a_i(\boldsymbol{x})$ posses order $\rho_i = 0$. Then, the equation $\mathcal{P}(x_1, \xi) = 0$ of the corresponding horizontal segment has solutions $c_k(x_2^{1/d})$ in the form of Puiseux series.*

Proof. The terms associated with the horizontal segment define points $\pi_\ell = (k, \rho_\ell^{(1)})$ belong to the set Π'' and are such that define a monic polynomial in α:

$$\mathcal{P}(\alpha) = \hat{a}_k(x_2)\alpha^m + \hat{a}_{k_1}(x_2)\alpha^{m-k_1} + \cdots + \hat{a}_{k_s}(x_2)\alpha^{m-k_s} = 0,$$

where $\hat{a}_i(x_2) = a_i(0, x_2)$. Now, by Puiseux theorem we know that the above equation posses k_s solutions $c_j(x_2)$ in the form of Puiseux series. □

Now, at the first step of the process, the case with a negative slope is considered.

Proposition 3.2.3. *Assume that f has the same structure than* (3.4) *and assume that the first Newton diagram posses a segment with negative slope. Then, there exist a change of variables $(z, x_1, x_2) \mapsto (\zeta, y_1, y_2)$ such that the polynomial $\mathcal{P}(y_2, \xi)$ has Puiseux series solutions $c_k(y_2^{1/d})$.*

Hence, applying to f the change of variables $z = \zeta$, $x_1 = y_1^{a_1}$ and $x_2 = y_2^{a_2}$ we get $\tilde{f}(\zeta, y_1, y_2)$, which can be solved. Therefore, solutions $z(x_1, x_2)$ of $f = 0$ are obtained by applying the inverse change of variables to solutions ζ. The following theorem allow us to use the iterative Newton procedure described above.

Theorem 3.2.1 (See [86].)**.** *The iteration of the classical Newton Procedure for one variable gives rise to representation of all the roots of the equation* (3.4) *by generalized Puiseux series with terms $\boldsymbol{x}^{\boldsymbol{a}/d}$, $d \in \mathbb{Z}_+$, such that $\boldsymbol{a}$ belong to n-dimensional, lex-positive strictly convex polyhedral cone.*

Proof. Let us choose a segment with slope $-\beta_t > 0$ and the set of points $\Pi^{(t)} \subset \Pi'$:

$$\{(k_1, \rho_{k_1}), (k_2, \rho_{k_2}), \ldots, (k_s, \rho_{k_s})\},$$

that belong to the segment. From this, we are able to obtain a polynomial equation $\mathcal{P}(\alpha, x_2) = 0$. In general, this polynomial is non-monic, and in order to regularize the Newton polygon, i.e., avoid horizontal segments in the subsequent steps and obtain a monic polynomial, we take into consideration the following change of variables:

$$z = \zeta \quad x_1 = v_1 x_2^a \qquad x_2 = v_2, \quad a \in \mathbb{N}.$$

Thus, the variable x_1 will remain of the same order and x_2 is increased by a. We can observe that ρ_k and the set $\Pi^{(t)}$ remains the same after this change of variables, thus the Newton polygon of $f(\zeta, v_1, v_2)$ does not change. Now, we look for the natural number a such that every coefficient in $f(\zeta, v_1, v_2)$ can be expressed as

$$a_k(v_1, v_2) = v_1^{\alpha_{k,1}} v_2^{\alpha_{k,2}} \hat{a}(v_1, v_2) \quad \text{with } \hat{a}(0,0) \neq 0.$$

After this change of variables, we are able to define the sets of integers

$$\begin{aligned}\mathcal{E}_1 &:= \{\rho_{t_1} - \rho_{t_2} = \alpha_{t_1} - \alpha_{t_2} : 1 \leq t_2 \leq t_1 \leq s\},\\ \mathcal{E}_2 &:= \{\alpha_{t_2,2} - \alpha_{t_1,2} : 1 \leq t_2 \leq t_1 \leq s\}.\end{aligned}$$

where ρ_k is the order of a_k (3.5). Now, in order to avoid the non-regular case in the following steps of the process, we find the positive integer e such that each element of $e\mathcal{E}_1$ is greater than elements of $\mathcal{E}_2$. With this, we can make the following change of variables

$$v_1 = y_1 v_2^e \qquad v_2 = y_2.$$

Thus, the exponents of the monomials $v_1^{\alpha_{t,1}} v_2^{\alpha_{t,2}}$ satisfy $e\,(\alpha_{\ell,1} - \alpha_{\ell,1}) > \alpha_{t_2,2} - \alpha_{t_1,2}$, implying that the following inequality holds

$$e\alpha_{t_2,1} + \alpha_{t_2,2} < e\alpha_{t_1,1} + \alpha_{t_1,2}. \tag{3.6}$$

Hence, the polynomial associated with the segment with a negative slope is given by

$$\mathcal{P}(\xi, y_2) = \hat{a}_{k_1}(y_2)\xi^{m-k_1} + \cdots + \hat{a}_{k_s}(y_2)\xi^{m-k_s},$$

where $\hat{a}_k(y_2) = y_1^{-\rho_1} a_k(y_1, y_2)$. By relation (3.6) we can see that w_{k_1} divides the polynomial P making it a monic polynomial and its k_s solutions are the coefficients of $x_1^{\beta_\ell}$. □

Remark 3.2.1. *The change of variables in the above proof can be expressed as a linear transformation defined by*

$$A = \begin{pmatrix} 1 & 0 \\ a+b & 1 \end{pmatrix} \tag{3.7}$$

acting on the vector of exponents $(a_1, a_2)^T$ of the monomials of $a_i(\boldsymbol{x})$. and then, solve the equation $f(\zeta, y_1, y_2)$.

Remark 3.2.2. *It is worth mentioning that the previous change of variables, can also be derived by introducing some $\mathbb{R}-$linear automorphisms, see, [86] for further details.*

3.2.3 Puiseux Series for Quasi-Polynomials with two delays

In what fallows it is assume that $(s^*, \tau_1^*, \tau_2^*) = (0, 0, 0)$ is a m-multiple root of the quasi-polynomial:

$$f(s, \tau_1, \tau_2) = p_0(s) + \sum_{i=1}^{2} p_i(s) e^{-s\tau_i}. \tag{3.8}$$

Proposition 3.2.4. *Consider the following quasi-polynomial*

$$f(s, \tau_1, \tau_2) = p_0(s) + p_1(s)e^{-s\tau_1} + p_1(s)e^{-s\tau_2},$$

with $s = 0$ a m-multiple root at $\boldsymbol{\tau} = (0, 0)$ and local representation $f(s, \boldsymbol{\tau}) = W(\boldsymbol{\tau})b(s, \boldsymbol{\tau})$. If $n_i^{(j)} = 0$ for $i = 0, 1, \ldots, k$ then, the $k + 1$ coefficients of the Weierstrass polynomial W satisfy

$$w_{m-i}(\boldsymbol{\tau}) \equiv 0, \qquad i \in \{0, 1, \ldots, k\}.$$

Proposition 3.2.5. *Let quasi-polynomial $f(s, \boldsymbol{\tau})$ have a m-multiple roots $s = 0$ at $\boldsymbol{\tau} = (0, 0)$, with associated Weierstrass polynomial W. Assume that*

$$\mathcal{R}\left(W, \frac{\partial W}{\partial s}\right) = \tau_1^{a_1}\tau_2^{a_2}\mathcal{U}(\tau_1, \tau_2) \quad \textit{such that}\, \mathcal{U}(0, 0) \neq 0,$$

where $(a_1, a_2) \in \mathbb{Z}_+^2 \setminus \{\mathbf{0}\}$, $\mathcal{U} \in \mathbb{C}\{\tau_1, \tau_2\}$. Then, $f = 0$ posses m solutions given by a generalized Puiseux series.

The following result gives conditions to have a regular Newton diagram.

Proposition 3.2.6. *Let $W(s, \tau_1, \tau_2)$ be the Weierstrass polynomial of a given quasi-polynomial $f(s, \tau_1, \tau_2)$. Assume that for a given $\ell-$segment of the Newton diagram, be $-\beta_\ell < 0$ its slope with corresponding points $\Pi^{(\ell)} = \{(k_1, \rho_{k_1}), (k_2, \rho_{k_2}), \ldots, (k_s, \rho_{k_s})\}$. Then, the equation $\mathcal{P}$ can be solved without any change of variables if the leading terms of the coefficients w_{k_i} satisfy*

$$w_{k_i,(\rho_{k_i},\eta_{k_i})} \neq 0 \textit{ whenever } \eta_{k_i} > \eta_{k_s}, \quad i < s.$$

.

Finally, using iterated Newton diagram procedure together with the Weierstrass polynomial of the quasi-polynomial $f(s,\boldsymbol{\tau})$, we can find the leading terms of the power series solutions.

Proposition 3.2.7. *Let $s^* = \boldsymbol{i}\omega^*$ be a $m-$multiple root of $f(s,\boldsymbol{\tau})$ at $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$. Assume that $\kappa = 0$ and r, β_j, (i_j, ℓ_j), m_j and $\Pi^{(j)}$, for $j = 0, 1, \ldots, r-1$ are given by Algorithm 1. Then, at $\boldsymbol{\tau} = \boldsymbol{\tau}^*$ the $m-$zeros of $f(s,\boldsymbol{\tau})$ can be expanded as*

$$s_{j\sigma}(\boldsymbol{\tau}) = \boldsymbol{i}\omega^* + c_{j\sigma}(\tau_2)\,(\tau_1 - \tau_1^*)^{\beta_j} + o\Big(|\tau_1 - \tau_1^*|^{\beta_j}\,|\tau_2 - \tau_2^*|^{\beta_j'}\Big)\,,$$

for $j = 0, 1, \ldots, r-1$, $\sigma = 0, \ldots, m_j$ and $m = m_0 + \cdots + m_{r-1}$. For $\beta_j > 0$ the coefficients $c_{j\sigma}(\tau_2)$ are roots of the polynomial:

$$\mathcal{P}_j(\xi,\tau_2) = \sum_{k=i_{j-1}}^{i_j} w_{k,(n_k^{(1)},n_k')}\tau_2^{n_k'}\xi^{k-i_{j-1}}, \quad \Big(k, n_k^{(1)}\Big) \in \Pi^{(j)},$$

when $\beta_{r-1} = 0$, the coefficients are given by the solution of

$$\mathcal{P}_j(\xi,\tau_2) = \sum_{k=i_{j-1}}^{i_j} w_{k,(0,n_k^{(2)})}\tau_2^{n_k^{(2)}}\xi^{k-i_{j-1}}, \quad (k, 0) \in \Pi^{(r-1)}$$

where $n_k^{(1)}$, $n_k^{(2)}$, n_k' are given by the first non-zero partial derivatives of Notation 3.2.1; the constant terms $w_{k,(n,\eta)} \in \mathbb{C}$ are computed using (3.3).

3.3 Double Root Characterization

It is known that some difficulties are present in the study of Puiseux series solutions (see, for instance, [50] for some insights). In some cases, such solutions do not always exist (see an example in subsection 3.1.1 as well as the discussion presented in [6, 7]). For instance, the migration of double characteristic root depending on two parameters is studied in [32, 42], where the authors have proposed two sectors to study the behavior of these roots when delays are subject to small deviations and restricted to such sectors. In this vein, it is worth mentioning that unlike the case of a single parameter, in the multi-parameter case there exist some singular and unexpected behaviors (see, for instance, the motivating example in 3.1.1) which have to be taken into account (see, for instance, [67]), in order to make the problem well-posed. In other words, the Puiseux type arguments cannot be extended straightforwardly from one parameter to a multi-parameter case.

3.3.1 Some Assumption

As mentioned in the introduction, this work is motivated by [42], where the authors analyzed the migration of a double root subject to perturbation of two parameters. In order to apply their approach, it is assumed that the characteristic function $f(s, x_1, x_2)$ satisfy the following assumption

$$\mathscr{D} = \det \begin{pmatrix} \Re\left(\frac{\partial f}{\partial x_1}\right) & \Re\left(\frac{\partial f}{\partial x_2}\right) \\ \Im\left(\frac{\partial f}{\partial x_1}\right) & \Im\left(\frac{\partial f}{\partial x_2}\right) \end{pmatrix} \neq 0. \tag{3.9}$$

If condition (3.9) is held, then f must possess non-zero partial derivatives and at least one must have a non-zero real and non-zero imaginary part. Thus the assumption limits the study to a particular case of polynomials.

Example 3.3.1. *In this example, we consider a polynomial borrowed from [42] that does not satisfy the condition* (3.9)*, given by*

$$f(s, x_1, x_2) = s^5 + s^4 + x_2 s^3 + (x_1 + 1)s^2 + s + x_1, \tag{3.10}$$

with a double root $s^* = \boldsymbol{i}$ at $(x_1, x_2) = (1, 2)$. In order to verify the condition, the first partial derivatives are computed:

$$\left.\frac{\partial f}{\partial x_1}\right|_{(\boldsymbol{i},1,2)} = 0$$
$$\left.\frac{\partial f}{\partial x_2}\right|_{(\boldsymbol{i},1,2)} = -\boldsymbol{i}.$$

Therefore $\mathscr{D} = 0$ and the assumption (3.9) *is violated. In Illustrative Examples 3.4 the root behavior is treated using our proposed approach based on the Weierstrass polynomial.*

Similar restrictions can appear when quasi-polynomials are considered.

Example 3.3.2. *Lets consider the quasi-polynomial:*

$$f(s, \boldsymbol{\tau}) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2} \tag{3.11}$$

where

$$\begin{aligned} p_0(s) &= s^5+s^4+\frac{4+\pi}{2}s^3+2s^2+\frac{2+\pi}{2}s+2, \\ p_1(s) &= 1, \quad p_2(s) = 2s^4+4s^2+2. \end{aligned}$$

For $(\tau_1, \tau_2) = (\pi, 1)$*,* f *has a double root at* $s = \boldsymbol{i}$*. Then the first non-zero partial derivatives of the quasi-polynomial at the double root are given by*

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{(0,\mathbf{0})} = \boldsymbol{i},$$
$$\left.\frac{\partial^n f}{\partial \tau_2^n}\right|_{(0,\mathbf{0})} = 0, \quad \forall n \in \mathbb{N}.$$

Since the condition (3.9) *is violated, in order to compute the asymptotic behavior of the double root in the following sections this condition will be relaxed, and based on the Weierstrass polynomial the root behavior will be treated in 3.4.*

Remark 3.3.1. *By changing the point of view in which the implicit function theorem is usually applied, is possible to represent the parameters* x_1, x_2 *as a function of* s*, mapping a neighborhood of* s_0 *in the complex plane to a neighborhood of the critical value* (x_1^*, x_2^*)*.*

In view of the *non-degeneracy* assumption (3.9) and the fulfillment of multiplicity for a double root ($m = 2$), the stability crossing curve can be found and together with conditions for the characterization of the splitting.
As mention in the Introduction, our approach is based on the properties of the associated Weierstrass polynomial (1.2). This allows the use of algebraic properties for the root behavior analysis. In the following we relax these conditions in order to consider a wider range quasi-polynomial; this within the scheme that we propose.

3.3.2 Regularity Condition

We now consider quasi-polynomials $f(s, \tau_1, \tau_2)$ (3.8) with a multiple root $s = 0$ at (τ_1^*, τ_2^*). Furthermore we restrict the analysis of f to satisfy the following assumption.

Assumption 3.3.1 (Regularity Condition)**.** *Let $s = \boldsymbol{i}\omega$ be a m-multiple of the quasi-polynomial f at (τ_1^*, τ_2^*), we will assume that*

$$\left.\frac{\partial f}{\partial \tau_i}\right|_{(0,\mathbf{0})} \neq 0. \tag{3.13}$$

for at least one τ_i with $i = 1, 2$.

We can see how the assumption above those given in (3.9), as in can be seen Example 3.3.1, where the condition that is fulfilled is (3.13).
In the following, we will return to the study of delay as a varied parameter, taking $p_i = \tau_i$.

Lemma 3.3.1. *The equation $f(s, \tau_1, \tau_2) = 0$ is satisfied by the unique solutions τ_1 and τ_2 defined by continuous functions $\tau_i(s_0)$ with $s_0 \in \mathcal{V}_\delta^*(\boldsymbol{i}\omega)$ excluding the double root $s^* = \boldsymbol{i}\omega$ if the following condition is satisfied:*

$$\left.\frac{\partial f}{\partial \tau_i}\right|_{s_0} \neq 0, \quad i = 1, 2.$$

3.3.3 Puiseux Series for Quasi-Polynomials with Two Delays

By means of a recursive procedure, the authors in [57] propose a method to compute the splitting behavior (see Section 2.4.2) of multiple roots for quasi-polynomials under the variation of a time-delay, in the commensurate delay case. This method is based on the corresponding partial derivatives and appropriate combinatorial factors related in some recursive way. This procedure can be extended to systems with two delay parameters, by defining an appropriate solution surface around multiple roots. In other words, the space curve $\mathcal{C}$ defined by the set $\{(s, \tau_1, \tau_2) \in \mathbb{C} \times \mathbb{R}_+^2 : f = 0\}$ can be parameterized by fractional power series in (τ_1, τ_2) called Puiseux series, as in the case of one parameter delay discussed in Chapter 2.

Proposition 3.3.1. *Let the regularity condition* (3.13) *holds for $i = 1$. Then, for a quasi-polynomial equation admits solutions in the form of Puiseux series in τ_1 in the following form*

$$s(\tau) = c_1(\tau_2)\tau_1^{1/m} + o\left(\tau_1^{1/m}\tau_2^{1/m}\right)$$

where the coefficients $c_k(x_2)$ can be expressed as a power fractional series in τ_2.

Proof. The coefficients $w_i(\tau_1, \tau)$ depend on derivatives of f as in equation by definition (3.3). We can see that the partial derivatives with respect to s $f_{i,\mathbf{0}}$ are:

$$p_{0,i}(s) + \sum_{k=1}^{2}\sum_{j=0}^{i}\binom{i}{j}(-1)^j\tau_k^j p_{k,(i-j)}(s)e^{-s\tau_k},$$

where the terms $p_{i,\ell}$ is the ℓ derivative of the polynomial $p_i(s)$. We can see that the polynomial structure is preserve until the $n-1$ partial derivative with respect to s. Now, the partial derivative f_{i,h_k} $(k = 1, 2)$ is given by

$$\sum_{\nu=0}^{1}\sum_{j=\nu}^{i}\binom{i}{j} j^{\nu}(-1)^{j+1-\nu}s^{1-\nu}\tau_k^{j-\nu}p_{k,(i-j)}(s)e^{-s\tau_k},$$

where $h_1 = (1, 0)$ and $h_2 = (0, 1)$. From this we can see that the partial derivatives with respect to τ_k depends on the polynomial p_k, delay τ_k and the partial derivatives $f_{i,(n,m)} = 0$ $\forall n, m \in \mathbb{N}$. The computation of the Puiseux series expansion was presented in [60], using the Newton Diagram method in an iterative procedure. $\square$

The above results give some explicit representation of m-solutions which determine the solution surface. These solutions are in the form of Puiseux series, which gives insights on the Splitting for fixed τ_2. In previous works, by using iterated Newton diagram procedure, the leading terms of the quasi-polynomial $f(s, \boldsymbol{\tau})$ are given in an explicit manner. Now, under the regularity condition (3.13) the following property guarantees the existence of Puiseux series solutions in the general case.

Proposition 3.3.2. *Suppose that the Assumption* (3.13) *is satisfied for τ_i with $i = 1, 2$. Then the leading terms are given as*

$$s_j(\boldsymbol{\tau}) = \boldsymbol{i}\omega^* + c_1\,(\tau_1 - \tau_1^*)^{1/m} + c_2\,(\tau_2 - \tau_2^*)^{1/\beta} + \tag{3.14}$$

$$o\Big(|\boldsymbol{\tau} - \boldsymbol{\tau}^*|^{1/m}\Big), \tag{3.15}$$

with $\beta \leq m$.

Remark 3.3.2. *Some methods for the computation of Puiseux series, using an extension of the Newton Diagram Method, known as* Polyhedral Algorithm *for* Puiseux expansions *(see, for instance, [3]). In this work we avoid this computation, more precisely, we focus on the implications of the Weierstrass preparation theorem in the existence of solutions from a different point of view.*

Proposition 3.3.3. *Suppose that the quasi-polynomial f condition of Proposition 3.3.1. If the following partial derivatives are satisfy*

$$\frac{\partial^n f}{\partial \tau_2^n} = 0, \quad \frac{\partial^{n+1} f}{\partial \tau_2^{n+1}} \neq 0, \tag{3.16}$$

then the root behavior is characterized by the partial multiplicities (m_1, m_2), and given explicitly by the Puiseux series solution

$$s_i(\tau_1, \tau_2) = \tau_1^{1/m_1} \tau_2^{1/n_2} \phi(\tau_1^{1/m_1}, \tau_1^{1/n_2})$$

such that $\sum m_i = m$, $n_i \leq m_i$ and $\phi(0,0) \neq 0$.

The above result is also satisfied when the delays are interchanged; assuming the regularity condition (3.13) for τ_2 and (3.16) for τ_1, and the multiple root is called *quasi-ordinary*.

3.3.4 Double Root of Quasi-Polynomials

The above construction does not give a characterization of the root behavior depending on the parameter space (x_1, x_2). With the aim of describing the behavior of the roots, we make explicit the relation between the local form (quadratic polynomial in s) and the branches of the double root. The quasi-polynomial $f(s, \tau_1, \tau_2)$ can be expressed by a quadratic polynomial in s with coefficients depending in (τ_1, τ_2), as follows

$$W(s, \tau) = s^2 + w_1(\tau_1, \tau_2)s + w_0(\tau_1, \tau_2). \tag{3.17}$$

It is possible to characterize the root behavior by means of the Discriminant in s D_s associated with W:

$$D_s(\tau_1, \tau_2) = w^2(\tau_1, \tau_2) - 4w_0(\tau_1, \tau_2). \tag{3.18}$$

In the space of delay parameters (τ_1, τ_2), the condition $D_s = 0$ guarantees the existence of a double root.

Proposition 3.3.4. *Let $s^* = \boldsymbol{i}\omega$ be a double root of the quasi-polynomial $f(s, \tau)$. Then there exists a change of variable such that the roots have the form:*

$$z^2 = \phi(\tau_1, \tau_2),$$

where ϕ can be expressed in the form convergent power series of order 1 in τ_1 and τ_2.

3.3.5 Complete Regular Splitting

In order to obtain information on the migration of a double root, we study the properties of the branch given Puiseux series without explicit computation. First, the condition for the existence of solutions defined in a neighborhood of $(0, 0)$, are given by Proposition 3.3.2. The migration of the double roots in all cases is summarized in the following proposition.

Proposition 3.3.5. *Suppose that f satisfies the condition $D_s = 0$, where D_s is given in* (3.18)*. If the Regularity condition* (3.13) *is satisfy for $x_1 = \tau_1$ and $x_2 = \tau_2$, then the two roots in a neighborhood of $\boldsymbol{i}\omega$ satisfy the asymptotic relations:*

$$s_{1,2}(\tau) = \pm c_1 \tau_1^{1/2} + \pm c_1 \tau_2^{1/2} + o(\tau_1 \tau_2),$$

Furthermore, if the partial derivatives satisfy

$$\Re\left(\frac{\partial f}{\partial \tau_i}\right) \neq 0, \quad i = 1, 2,$$

we say that the solution posses the Completely Regular Splitting Property (CRS).

3.4 Illustrative Examples

In this section, we consider several numerical examples encountered in the control literature, that will allow us to illustrate the effectiveness of the proposed results. As in the previous chapter, the numerical computation have been performed using the software package DDE-BIFTOOL.

Example 3.4.1. *We continue with the asymptotic analysis of the double root of the polynomial* (3.10)*. By the Weierstrass Preparation Theorem, the local form is given by its Weierstrass polynomial* $W(s, x_1, x_2) = (s - \boldsymbol{i})^2 + w_1(x_1, x_2)(s - \boldsymbol{i}) + w_1(x_1, x_2)$*, as discussed in Section 3.3.4. Following Proposition 3.3.5 the roots can be expressed as Puiseux series solutions as follows*

$$s_{1,2}(x_1, x_2) = \boldsymbol{i} \pm \frac{1 + \boldsymbol{i}}{\sqrt{2}}(x_1 - 1)^{1/2} + o\left(|x_1 - 1|^{1/2}|x_2 - 2|^{1/2}\right),$$

describing the splitting of the double root as shown in figure 3.3.

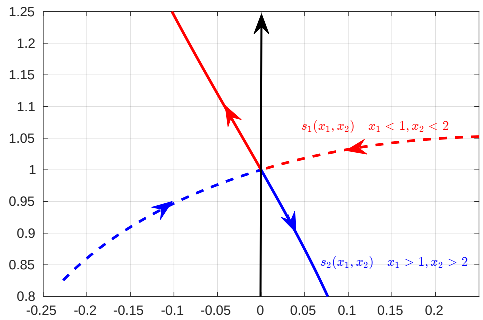


Figure 3.3: Root locus of quasi-polynomial $f(s, \mathbf{x})$ (3.10) around the double root $(\boldsymbol{i}, 1, 2)$.

Example 3.4.2. *Consider the following quasi-polynomial*

$$f(s, \boldsymbol{\tau}) = \left(s^2 - 2s + 2\right) + \left[2\cos(1)s - 2\left(\cos(1) + \sin(1)\right)\right]e^{-\tau_1 s} + e^{-\tau_2 s}. \quad (3.19)$$

Simple computations show that for $(\tau_1, \tau_2) = (1, 2)$*,* f *has a critical root at* $s^* = \boldsymbol{i}$ *with multiplicity two. Additionally, the first partial derivatives are given by*

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{(\boldsymbol{i},1,2)} \approx 2.91 + \boldsymbol{i}0.584, \qquad \left.\frac{\partial f}{\partial \tau_2}\right|_{(\boldsymbol{i},1,2)} \approx -0.91 + \boldsymbol{i}0.416.$$

Applying Proposition 3.3.1, we conclude that the solutions of the Weierstrass polynomial can be expanded as a Puiseux series:

$$\begin{aligned} s_{1,2}(s, \boldsymbol{\tau}) = \boldsymbol{i} &\pm (1.659 + \boldsymbol{i}0.755)\left(\tau_1 - \tau_1^*\right)^{1/2} \\ &\pm (1.058 - \boldsymbol{i}1.051)\left(\tau_2 - \tau_2^*\right)^{1/2} + o\left(|\tau_1 - 1|^{1/2}|\tau_2 - 2|^{1/2}\right). \end{aligned}$$

For fixed $\tau_2 = 2$*, the splitting is shown in figure 3.4.*

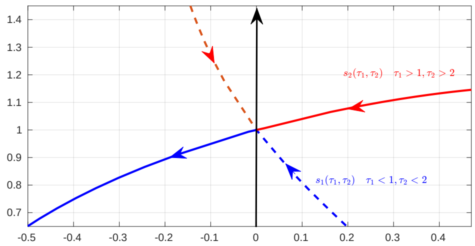


Figure 3.4: Root locus of quasi-polynomial $f\,(s, \boldsymbol{\tau})$(3.19) around the double root $(\boldsymbol{i}, 1, 2)$.

Example 3.4.3. *Now we continue with the quasi-polynomial treated 3.3.2,* $f(s, \boldsymbol{\tau}) = p_0(s) + p_1(s)e^{-s\tau_1} + p_2(s)e^{-s\tau_2}$*, with partial derivatives:*

$$\left.\frac{\partial f}{\partial \tau_1}\right|_{(0,\mathbf{0})} = \boldsymbol{i}, \qquad \left.\frac{\partial^n f}{\partial \tau_2^n}\right|_{(0,\mathbf{0})} = 0, \ \forall n \in \mathbb{N}, \tag{3.20}$$

$$\left.\frac{\partial^2 f}{\partial s \partial \tau_1}\right|_{(0,\mathbf{0})} = 1 - \boldsymbol{i}\pi, \qquad \left.\frac{\partial^{n+1} f}{\partial s \partial \tau_2^n}\right|_{(0,\mathbf{0})} = 0, \ \forall n \in \mathbb{N}. \tag{3.21}$$

Since the critical roots are double the associated Weierstrass polynomial W *has the following structure:*

$$W(s, \tau) := s^2 + w_1(\boldsymbol{\tau}) + w_0(\boldsymbol{\tau}).$$

Moreover, since the regularity condition (3.13) *holds, we only need to compute* $w_0(\boldsymbol{\tau})$*. After simple computations, one gets*

$$w_0(\boldsymbol{\tau}) = \frac{-2\boldsymbol{i}}{(8 + \pi^2) + \boldsymbol{i}(8 - 3\pi) + 16e^{-\boldsymbol{i}}}\tau_1 + \mathcal{O}\left(\tau\right).$$

Thus, its solutions are given by:

$$s(\boldsymbol{\tau}) = \boldsymbol{i} \pm \frac{\sqrt{2}\boldsymbol{i}^{3/2}}{\sqrt{(8+\pi^2)+\boldsymbol{i}(8-3\pi)+16e^{-\boldsymbol{i}}}}(\tau_1 - \pi)^{1/2} + \mathcal{O}\left(|\boldsymbol{\tau} - \tau^*|\right).$$

Proposition 3.3.5 guarantees that the solution $(\boldsymbol{i}, \pi, 1)$ *has the CRS property. This behavior is illustrated in Figure 3.5.*

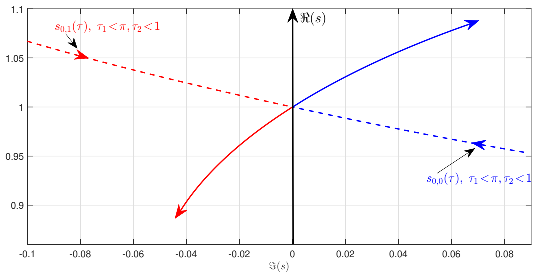


Figure 3.5: Root locus of quasi-polynomial $f(s, \boldsymbol{\tau})$ (3.11) around $(\boldsymbol{i}, \pi, 1)$.

3.5 Chapter Summary

In this chapter, we have considered some issues concerning the asymptotic behavior of multiple critical roots for quasi-polynomials with two delays. By means of the Weierstrass polynomial, the proposed approach is based on an iterative Newton diagram method which can be effectively applied to find the leading terms of power series solutions expressed as a generalized Puiseux series. Finally, the splitting properties of a given solution have been described by means of CRS, RS, and NRS properties in order to get some insights into its geometric behavior.

Chapter 4

Regular and Singular Dependence on the Time-Delay

This chapter deals with some singular problems that arise when implementing a conventional PD-controller by means of a delay-difference scheme. The main goal will be to derive a simple methodology to characterize some ill-posed cases related to such singular problems, which, to the best of the author's knowledge is a problem still open in the control literature. In order to accomplish such a task, convergent series will be the main tool to express the singular solutions. More precisely, by means of an appropriate change of variable, the expansion of unbounded solutions will be obtained, which allows deriving an appropriate auxiliary solution that serves to analyze the behavior of such solutions.

4.1 Approximation of the Derivative Action

In this chapter, the following strictly proper Linear Time-Invariant SISO systems Σ will considered, with state-space representation:

$$\Sigma := \begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t) \end{cases} \tag{4.1}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c^T \in \mathbb{R}^n$. The related transfer function H_{yu} of Σ is given by

$$H_{yu}(s) := c^T (sI - A)^{-1} b = \frac{P(s)}{Q(s)}, \tag{4.2}$$

with

$$Q(s) = s^n + \sum_{i=0}^{n-1} q_i s^i, \qquad P(s) = \sum_{i=0}^{m} p_i s^i, \qquad m > n.$$

Employing the conventional output PD-feedback control law

$$u(t) = -k_p y(t) - k_d \dot{y}(t),$$

where $(k_p, k_d) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we get the controller C_0 given by

$$C_0(s) = k_p + k_d s. \tag{4.3}$$

Additionally, denote by $Stab(\Sigma)$ the class of stabilizing controllers of (4.1).
One of the simplest ways to implement a PD controller is to use the *delay-difference operator* to approximate the derivative term. Thus, for small delay values, the following approximation (standard Euler approximation) given by

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau},$$

is considered on the control law $u(t)$, to obtain the controller:

$$C_\tau(s) = k_p + k_d \frac{1 - e^{-s\tau}}{\tau}. \tag{4.4}$$

From the above definitions, it is easy to see that when $\tau \to 0^+$ we have that $C_\tau \to C_0$. Such arguments could indicate that this approximation preserves the C_0 controller properties, however, it is worth emphasizing that, under some choice of the control parameters (k_p, k_d), such an argument does not necessarily hold during the transition from the controller C_τ to C_0. As will be seen in the motivational example in Section 4.3, we may find situations in which unexpected dynamics behaviors are present.
The objective of this chapter is to further explore such a problem from the asymptotic stability point of view and to characterize the existing links between C_0, C_τ and $Stab(H_{yu})$. In other words, conditions will be sought in which C_0 and C_τ are both in $Stab(H_{yu})$ as well as the limit cases when C_0 and C_τ are not simultaneously in $Stab(H_{yu})$. To the best of the author's knowledge, such a problem has not been fully addressed in the literature and the existing results do not allow to have a complete characterization of the corresponding case study.

In this chapter, a change is made in the notation of the characteristic equation. In what follows the controller characteristic quasi-polynomial is denoted by $\Delta(s,\tau)$ instead of $f(s,\tau)$ (as in the previous chapters), this due to the motivation of the chapter, which arises from a control system.

It is worth mentioning that these problems may be reformulated as analyzing the existing links between the spectral abscissa[1] of the characteristic quasi-polynomial $\Delta(s,\tau)$

$$\Delta(s;\tau) = Q(s) + \left(k_p + k_d \frac{1-e^{-s\tau}}{\tau}\right) P(s), \tag{4.5}$$

of the closed-loop systems involving C_τ for small delay values in comparison the characteristic polynomial $\Delta_0(s)$:

$$\Delta_0(s) = Q(s) + (k_p + k_d s)\, P(s), \tag{4.6}$$

with C_0. In addition, if $\tau \to 0^+$, we have that:

$$\frac{1-e^{-s\tau}}{\tau} = s - \frac{\tau}{2!}s^2 + \frac{\tau^2}{3!}s^3 - \frac{\tau^3}{4!}s^4 + \cdots + (-1)^j \frac{\tau^j}{(j+1)!}s^{j+1} + \cdots = \tag{4.7}$$

$$= s - \left(\underbrace{\sum_{j=2}^{\infty} (-1)^j \frac{\tau^{j-1}}{(j)!} s^j}_{=:\widehat{\Delta}_\tau(s)}\right). \tag{4.8}$$

Hence, it is easy to see that the quasi-polynomial Δ can be rewritten as follows:

$$\Delta(s;\tau) = \Delta_0\,(s) - k_d \widehat{\Delta}_\tau\,(s)\, P\,(s)\,. \tag{4.9}$$

Denote by $\sigma(\Delta)$ ($\sigma(\Delta_0)$) the spectrum of the quasi-polynomial Δ (polynomial Δ_0). The problem mentioned above is to understand the mechanism that makes the set $\sigma(\Delta)\cap\mathbb{C}_+$ not empty for infinitesimal delay values τ for a pair (k_p, k_d) of stabilizing gains ($C_0 \in Stab(H_{yu})$), that is $\sigma(\Delta_0) \subset \mathbb{C}_-$.

As shown in [84], even in the scalar case, a particular choice of parameters may lead to *strong sensitivity to delay parameters* in the sense that there exist

[1]the real part of the rightmost root of the corresponding characteristic function[66]

characteristic roots with arbitrarily large modulus in $\mathbb{C}_+$ for arbitrarily small delay parameters τ. It is worth mentioning that there exist situations where the use of a delay in the controller (see, for instance, [78, 36]) stabilize the system. In Section 4.6, explicit conditions will be derived, explaining the delay sensitivity mechanism in using the delay-difference approximation for the derivative action.

4.2 Ill-Posed Linear Systems

Ill-posed problems (see Section 4.4) for delay-differential equations have been previously reported in the literature, see, for instance, [28, 68, 53, 54, 29, 64]. In addition, the contributions of [53, 64] concern the sensitivity of the characteristic roots of the delay-difference operator associated with some neutral-type delay-differential systems. This sensitivity to small delays (lack of robustness) is also encountered when controlling some damped wave equations by using delays in the boundary control law (see, for instance, [28, 68]). Next, [29] focuses on some limitations of the control laws if implemented via the numerical quadrature still leading to the sensitivity of the characteristic roots of some delay-difference operator for neutral-type systems. Using different frequency-domain arguments, [54] deals with the sensitivity of the delay rays with respect to small perturbation on the delay ratio. Although the sensitivity of the closed-loop scheme with respect to small delay values was discussed in [53], it is worth mentioning that all the results mentioned above concern systems not possessing delay-dependent parameters. Finally, the stability of the system $C_\tau H_{yu}$ in closed-loop was discussed [44] by extended the so-called τ-decomposition method [48] to delay systems with delay-dependent parameters, focusing more on characterizing the exponential stability with some pre-specified decay rate without any deeper discussion of the case when $\tau \to 0^+$.

4.3 Problem Statement

The main goal of this section is to show the root sensitivity when the derivative action is approximate using delay. As mentioned earlier, there exist situations when the stability of the closed-loop of the system is affected by small delay values, however, when implemented through the delay-difference approximation, the closed-loop system may become unstable. Before continuing with the discussion, below is an example showing the subsequent development of this chapter.

Example 4.3.1. *Let's consider the following LTI SISO scalar system:*

$$\dot{y}(t) = p_0 y(t) + u(t), \tag{4.10}$$

where $p_0 \in \mathbb{R}_+$. *Then the ideal* proportional-derivative *control law given by:*

$$u(t) := k_p y(t) + k_d \frac{d}{dt} y(t), \tag{4.11}$$

where $(k_p, k_d) \in \mathbb{R}^2 \setminus \{(0,0)\}$.*Thus, the associated characteristic function* Δ_0 *can be written as:*

$$\Delta_0(s) := (k_d - 1)s + (k_p + p_0). \tag{4.12}$$

Since the only root s^* *of* (4.12) *is given by* $s^* = -(k_p + p_0)/(k_d - 1)$, *it follows that* (4.10) *will be exponentially stable in closed-loop iff* $(k_d - 1)(k_p + k_0) > 0$. *In particular, such a property holds for all pairs* (K_p, k_d) *satisfying* $k_d > 1$ *and* $k_p > -p_0$. *Now, if we use the delay-difference approximation for the derivative action, then the corresponding control law rewrites as:*

$$u(t) := k_p y(t) + k_d \frac{y(t) - y(t-\tau)}{\tau}, \tag{4.13}$$

with $\tau > 0$, *but sufficiently "small" leading to the characteristic function* $\Delta : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}$, *given by:*

$$\Delta(s;\tau) := (s - p_0) - \left(k_d \frac{1 - e^{-\tau s}}{\tau} + k_p \right). \tag{4.14}$$

It is easy to see that $\Delta(s;\tau) = s f(s;\tau)$, *where* $f : \mathbb{C}^* \times \mathbb{R}_+ \mapsto \mathbb{C}$ *is given by:*

$$f(s;\tau) := f_0(s) - f_\tau(s\tau), \tag{4.15}$$

where the functions $f_0 : \mathbb{C}^* \mapsto \mathbb{C}$ *and* $f_\tau : \mathbb{C}^* \mapsto \mathbb{C}$ *are defined as follows:*

$$f_0(z) := 1 - \frac{k_p + p_0}{z}, \qquad f_\tau(z) := k_d \frac{1 - e^{-z}}{z}.$$

From the above definitions, for $s \in \mathbb{C} \setminus \{0\}$, *both functions* f *and* Δ *share the same roots. Assume now that* $\tau > 0$, *along with the following conditions:*

$$k_d > 1, \qquad k_p + p_0 > 0. \tag{4.16}$$

First, observe that, under the restrictions (4.16)*, the closed-loop system is exponentially stable with the ideal PD-controller, i.e., for* $\tau = 0$*. Second, consider some real* $\xi \in \mathbb{R}_+$*. It is easy to see that* $f_0(\xi)$ *is an increasing function and for a fixed* $\tau > 0$*,* $f_\tau(\xi\tau)$ *is decreasing with respect to* $\xi\tau$*. The behaviors of* f_τ *and* f_0 *are illustrated in Fig 4.1. Furthermore, the following inequalities hold:*

$$-\infty < f_0(\xi) \leq 1, \quad \forall \xi \in \mathbb{R}_+ \quad \text{and} \quad 0 < f_\tau(\xi\tau) < k_d, \quad \forall \xi \in \mathbb{R}_+.$$

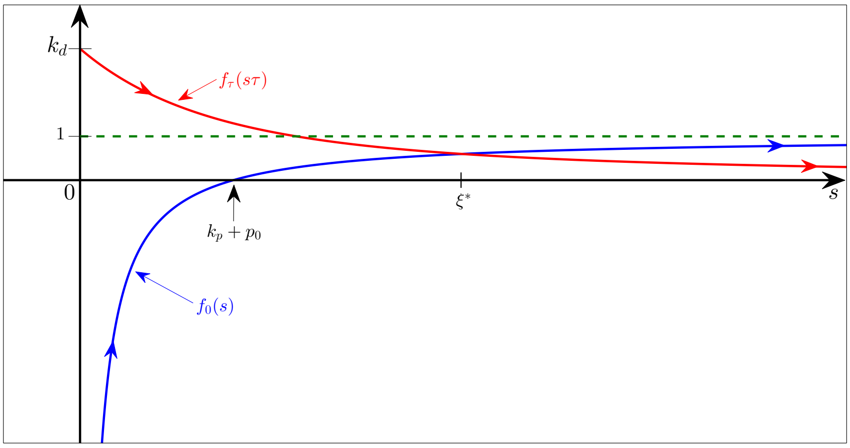


Figure 4.1: Behavior of f_τ and f_0 for $s \in \mathbb{R}_+$.

From the above discussion, it is easy to see that, for a fixed delay value $\tau > 0$*, under the assumptions* (4.16)*, the characteristic function* Δ *has always a positive root* $\xi^* \in \mathbb{R}_+$*. Moreover, since both* $f_\tau(\xi\tau)$ *and* $f_0(\xi)$ *are injective (with respect to* ξ*) functions, the positive solution* ξ^* *is* unique.

Finally, as we will discuss deeply in the next section, the rightmost root of Δ *behaves as:*

$$s(\tau) = \frac{z_0}{\tau} + \mathcal{O}(1), \tag{4.17}$$

where $z_0 \in \mathbb{R}_+$ *is given by* $z_0 = k_d + \mathcal{W}\left(-k_d e^{-k_d}\right)$*. Thus, according to* (4.17)*, as* $\tau \to 0^+$ *the real part of the roots of* Δ *will tend to* $\Re\{s\} \to +\infty$*. In particular, for* $k_d = 6 \Rightarrow z_0 \approx 5.984901226$*.*

The previous example shows the existence of a singular root of the characteristic function f of a scalar system for small delay values. Before characterizing this phenomenon, we proceed to define the points where it occurs.

4.3.1 Regular and Singular Points

In this section, the concept of a singular point of algebraic equations given in Section 1.2 is extended. From the context of Ordinary Linear Differential Equations we take the concept of ordinary and regular singular points (see, for instance, [11, 92]). To this purpose, the scalar parameter $x \in \mathcal{D} \subset \mathbb{C}$ is considered as the variable under analysis such that satisfies the following definitions.

Definition 4.3.1. *Let $\mathcal{D} \subset \mathbb{R}$ be an open subset, $p_0 \in \mathcal{D} \cup \partial\mathcal{D}$, $r \in \mathbb{N}$ and $g : \mathcal{D} \to \mathbb{C}$. Then, we say that a point p_0 is a singular point, if there exists some neighborhood $\mathcal{V}(p_0) \subset \mathcal{D}$ such that $g(p)$ is bounded $\forall p \in \mathcal{V}(p_0) \setminus \{p_0\}$. Moreover, the point p_0 is said to be a singular point of $rth-$order, if the following limit*

$$\lim_{p \to p_0} (p - p_0)^r \, g(p),$$

is finite.
Otherwise, the point p_0 is called a regular point.

From the previous definition, it is clear to see that in the case of having a singular point of rth-order in a neighborhood $\mathcal{V}(p_0)$ of p_0, g behaves as:

$$g(p) = \frac{z_0}{(p - p_0)^r} + \mathcal{O}(1),$$

for some $z_0 \in \mathbb{C}$.

Remark 4.3.1 (First-Order Singular Points)**.** *Following Definition 4.3.1, we say that 0_+ is a Regular Singular point of* first order *of the solution $s = \phi(\tau)$ if*

$$\lim_{\tau \to 0^+} \tau \phi(\tau)$$

exists and is finite.

The main goal is to give conditions in the parameters (k_p, k_d, τ) over the qualitative behavior of the solutions. In this vein, we first analyze the *well-posedness* of the closed-loop system $H_{yu}C_\tau$ with respect to the delay.

4.4 Ill-Posed/Well-Posed Approximation Laws

In the sequel, we will use the following notions:

Definition 4.4.1 (Ill-posed approximation law/ill-posed closed-loop system)**.** *Consider the LTI SISO system* (4.1) *with the transfer function* H_{yu} (4.2)*. Assume that the stabilizing controller* C_0 *(*$C_0 \in Stab(H_{yu})$*) in* (4.3) *is replaced by the approximation controller* C_τ *given by* (4.4)*. If there exists a sequence of positive real numbers* $(\tau_n)_{n\in\mathbb{N}}$ *satisfying the condition* $C_{\tau_n} \to C_0$ *where* $\tau_n \to 0^+$ *when* $n \to +\infty$ *such that* $C_{\tau_n} \notin Stab(H_{yu}$*, then the closed-loop system* (4.1)-(4.4) *is* ill-posed*. In this case, the controller* C_τ *is called an* ill-posed approximation controller*.*

Let us explain the notion above. First, it is easy to observe that since C_0 is a stabilizing controller, all the zeros of the characteristic function Δ_0 are stable characteristic roots, i.e. $\sigma(\Delta_0) \subset \mathbb{C}_-$. Now if a sequence of positive real numbers $(\tau_n)_{n\in\mathbb{N}}$ verifies the properties of the Definition 4.4.1, then it follows that $\sigma(\Delta) \cap C_+ \neq \emptyset$ for some infinitesimal small delays although $C_{\tau_n} \to C_0$ and C_0 is a stabilizing controller. More precisely, the continuity of the rightmost root (spectral abscissa)[2] wrt "infinitesimal small" delays is lost although such a (continuity) property holds for the corresponding controllers ($C_{\tau_n} \to C_0$ when $n \to +\infty$). In other words, the solutions of the closed-loop system becomes unbounded for some infinitesimal small delay values although such solutions were asymptotically stable for the closed-loop system free of delay $tau = 0$.

With the remarks above, the concept of *well-posed* closed-loop system and *well-posed* approximation law follow straightforwardly:

Definition 4.4.2 (Well-posed approximation law/well-posed closed-loop system)**.** *Consider the LTI SISO system* (4.1) *with the transfer function* H_{yu} (4.2)*. Assume that the stabilizing controller* C_0 *(*$C_0 \in Stab(H_{yu})$*) in* (4.3) *is replaced by the approximation controller* C_τ *given by* (4.4)*. If for any sequence of positive real numbers* $(\tau_n)_{n\in\mathbb{N}}$ *satisfying the condition* $C_{\tau_n} \to C_0$ *where* $\tau_n \to 0^+$ *when* $n \to +\infty$*, and for all* $\varepsilon > 0$*, there exists some* $N = N_\varepsilon \in \mathbb{N}$*, such that for all* $n \geq N_\varepsilon$*,*

[2] For a deeper discussion on the continuity properties of the spectral abscissa function wrt its parameters see, for instance, [66] and the references therein.

$C_{\tau_n} \in Stab(H_{yu}$, *then the closed-loop system* (4.1)-(4.4) *is* well-posed. *In this case, the controller* C_τ *is called an* well-posed approximation controller.

As expected, the well-posed case corresponds to the situation when the closed-loop system stays stable for some infinitesimal delay bounds. In the following section, we analyzed the effect of small delays on the derivative approximation scheme. This is done by proposing an appropriate change of variable allowing the analysis of the root behavior.

4.5 An Auxiliary Scalar Delay System

Consider now the following scalar delay-differential system:

$$\dot{y}(t) + \alpha\,(y(t) - y(t-1)) = 0, \tag{4.18}$$

under appropriate initial conditions, where $\alpha \in \mathbb{R}$ is a parameter. The corresponding characteristic function writes as $\Delta : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$:

$$\Delta(s, \alpha) := s + \alpha\left(1 - e^{-s}\right). \tag{4.19}$$

It is easy to see that $\Delta(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, showing that such a root is *invariant* with respect to α. Furthermore, since $d/ds\,(\Delta(s, \alpha)) = 1 + \alpha e^{-s}$, then the root at the origin is *double* if $\alpha = -1$, as depicted in Fig 4.2. With these observations in mind, we have the following first result:

Proposition 4.5.1. *(Characteristic roots on the imaginary axis) If the characteristic function* (4.19) *has roots on the imaginary axis, they are located at the origin.*

Proof. Indeed, let us proceed by contradiction. Let assume that, for some $\omega > 0$ $s = \boldsymbol{i}\omega$ is a characteristic root of (4.19). Then ω should satisfy the following conditions simultaneously:

$$\begin{cases} \omega + \alpha\sin(\omega) = 0 \\ cos(\omega) = 1. \end{cases} \tag{4.20}$$

Or these conditions hold if and only is $\omega = 0$ fact that contradicts our assumption. □

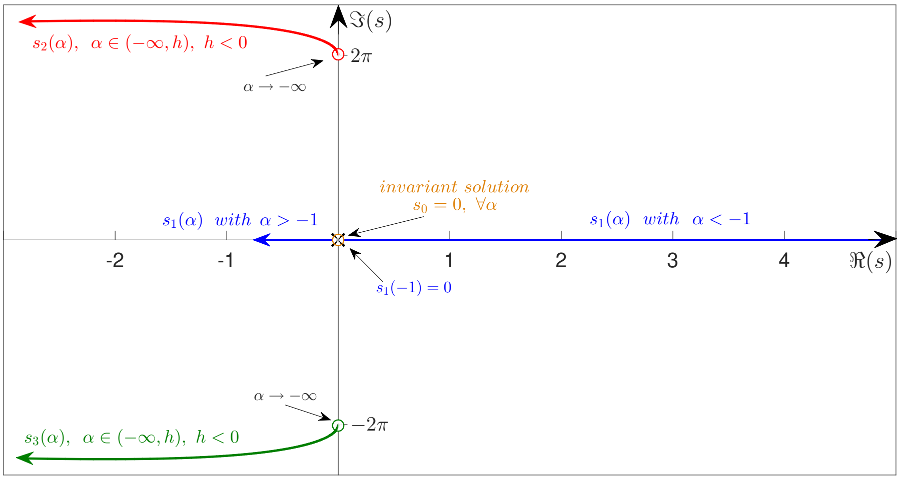


Figure 4.2: Root behavior of Δ_α.

In other words, the result above simply shows that if characteristic roots are crossing the imaginary axis they should cross through the origin. Based on such a property, we have the following lemma:

Lemma 4.5.1. *For all $\alpha \in (-\infty, -1)$, the scalar system (4.18) is unstable. Furthermore, the following properties hold:*

(i) *the characteristic function Δ has a strictly positive root;*

(ii) $\mathrm{card}\,(\sigma(\Delta) \cap \mathbb{C}_+) = 1$.

If $\alpha \in (-1, +\infty)$, excepting the root at the origin, the remaining characteristic roots of Δ_α (if any) are located in $\mathbb{C}_-$.

The Lemma above simply characterizes the stability of the trivial solution with respect to the parameter α. The case when $\alpha \in (-\infty, -1)$ illustrates some interesting properties:

(a) First, when the real parameter α is increased from $-\infty$, one real characteristic root arrives from $+\infty$ and it will move on the real axis towards to $-\infty$ when α tends to $+\infty$;

(b) Second, excepting the root crossing at the origin, there are no other characteristic roots crossing the imaginary axis wrt the parameter α.

We will call such a mechanism *"locking real unstable roots"*.

Finally, the existence of a characteristic root of on the real axis independently of the values taken by the real parameter α will play an important role in characterizing the *ill-posed* approximation scheme. More precisely, such roots will define the asymptotic behavior of the characteristic roots of $\Delta_\tau(s)$ for "small" delay values.

Remark 4.5.1. *It is worth mentioning that the stability of the trivial solution of the scalar delay-differential equation is well-known and it has received a complete characterization in the literature in the corresponding parameter-space and (see, for instance, Hayes [35], Bellman and Cooke [10], Hale and Verduyn Lunel [34], Michiels and Niculescu [66] and the references therein). To the best of the authors' knowledge, the results proposed by Hayes [35] give a complete characterization of the problem in both retarded and neutral cases.*

Proof. To show that as long as $\alpha \in (-1, +\infty)$, all the characteristic roots are located in $\mathbb{C}_-$, we will proceed by contradiction. Let $\lambda = r + \boldsymbol{i}\omega \neq 0$ be some non-zero root of (4.19), that is $\Delta(r + \boldsymbol{i}\omega, \alpha) = 0$, with $r \geq 0$. With no loss of generality, assume that $\omega \in \mathbb{R}_+$. Then, by evaluating $\Re(\Delta(\lambda, \alpha))$ and $\Im(\Delta(\lambda, \alpha))$ for $\lambda = r + \boldsymbol{i}$, it is easy to see that for all $(r, \alpha) \in \mathbb{R}_+ \times [1, +\infty)$,

$$\Re(\Delta(\lambda, \alpha)) := r + \alpha(1 - e^{-r}\cos(\omega)) \geq r + 1 - e^{-r} > 0. \tag{4.21}$$

Thus, the only remaining possibility is $r = 0$. Or, in this case, $\Re(\Delta(\boldsymbol{i}\omega)) = \alpha(1 - \cos(\omega)) = 0$ and $\Im(\Delta_\alpha(\boldsymbol{i}\omega)) = \omega + \alpha\sin(\omega) = 0$ if and only if $\omega = 0$, fact that contradicts our assumption that the characteristic root is non-zero. Next, if $\alpha \in (-1, 1)$, then for all $(r, \omega) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\Im(\Delta_\alpha(\lambda)) = \omega + \alpha e^{-r}\sin(\omega), \tag{4.22}$$

leading to the another contradiction since, for $\alpha \neq 0$ in the equality $e^r / \mid \alpha \mid = \mid \sin(\omega)/\omega \mid$, the left-term is greater than 1 ($e^r / \mid \alpha \mid > 1$) and the right-term is less than 1 ($\mid \sin(\omega)/\omega \mid < 1$), for all $(r, \omega, \alpha) \in \mathbb{R}_+ \times \mathbb{R}_+ \times (-1, 1)$, with $\alpha \neq 0$. Now, if $\alpha = 0$, then $\Im(\Delta_\alpha(\lambda)) = \omega \neq 0$. Thus, in conclusion, for all $\alpha \in (-1, +\infty)$, there are no unstable characteristic roots.

Consider now the case $\alpha \in (-\infty, -1)$. First of all, it is easy to see that the function $\Delta_\alpha : \mathbb{R}_+ \mapsto \mathbb{R}$ has only one root in $\mathbb{R}_+$. More precisely, Δ_α is an analytic convex function. Indeed, its first- and second-order derivative write as $\Delta'_\alpha(r) = 1+\alpha e^{-r}$ and $\Delta''_\alpha(r) = -\alpha e^{-r}$, respectively. Next, it is easy to observe that $r_{min} = \log(-\alpha) \in \mathbb{R}_+$ is the unique solution of the equation $\Delta'_\alpha(r) = 1 + \alpha e^{-r} = 0$ and that $\Delta_\alpha(r_{min}) < 0$ for all $\alpha \in \mathbb{R}_+$. Next, since $\Delta_\alpha(0) = 0$, there are no any other positive real roots for all $r \in (0, r_{min}]$. Finally, since the function is strictly increasing on the interval $(r_{min}, +\infty)$ and since $\Delta_\alpha(r_{min})\Delta_\alpha(+\infty) < 0$, then there exists a *unique* positive characteristic root $r^* \in (r_{min}, +\infty)$. The use of Proposition 4.5.1 ends the proof. □

The characterization of the scalar system discussed in this section will be very useful in the following asymptotic analysis.

4.6 Asymptotic Behavior Analysis

As seen in the motivational example 4.3, the delay-difference approximation may be ill-posed. Such behavior is the result of the existence of a chain of roots with a positive real part. In this section, the root behavior for small delay will be described by means of its asymptotic behavior. Inspired by the approach taken in [83], an adequate solution is sought using an appropriate change of variable to compute roots with singular behavior in the form of power series.

Lemma 4.6.1. *Consider the closed-loop system* (4.1)-(4.4) *with the corresponding characteristic function* $\Delta(\cdot, \cdot)$ *given as in* (4.5).

Let $(\tau_j) \in \mathbb{R}_+$, $j \in \mathbb{N}$ *be a sequence of positive real numbers. Then all the characteristic roots* $s(\cdot) : \mathbb{R}_+ \mapsto \mathbb{C}$ *of the function* $\Delta(\cdot, \cdot)$, *defined as functions of* $\tau_j \in \mathbb{R}_+$, *have the following properties:*

(i) $\tau_j s(\tau_j)$ *are bounded;*

(ii) *as* $\tau_j \to 0^+$ $\tau_j s(\tau_j) \to 0$ *or* $\tau_j s(\tau_j) \to z_j$ *for some complex* $z_j \in \mathbb{C}\backslash\{0\}$.

Proof. First of all, assume that $s(\tau) = \tau^{-\beta} w(\tau)$ is a root of Δ, such that $w(0) \neq 0$ and $\beta \in \overline{\mathbb{Q}}_+$. With these notations, define $\widetilde{\Delta}$ as follows:

$$\widetilde{\Delta}(w; \tau) := \tau^\alpha \Delta(\tau^{-\beta} w; \tau), \tag{4.23}$$

where $\alpha \in \overline{\mathbb{Q}}_+$. From (4.23), it is easy to see that both Δ and $\widetilde{\Delta}$ share the same solutions. Thus, in the remaining part of the proof, we will analyze the solution of $\widetilde{\Delta}$ for appropriate α and β. Now, in order to find the dominant part of $\widetilde{\Delta}$ when $\tau \to 0^+$, we will express $\widetilde{\Delta}$ in the following way:

$$\widetilde{\Delta}(w;\tau) = D(w) + R(w,\tau), \tag{4.24}$$

such that the functions D and R are analytic in $s \in \mathbb{C}$ and $(s,\tau) \in \mathbb{C} \times \overline{\mathbb{R}}_+$, respectively with the following properties:

(a) the function D does not depend on τ and there exists $w^* \in \mathbb{C} \setminus \{0\}$, such that $D(w^*) = 0$;

(b) the function R satisfies $\lim_{\tau \to 0^+} R(w,\tau) = 0$.

Now, as a first step, consider the case $\beta \equiv 0$. Such a condition clearly implies that $\alpha \equiv 0$. Hence, in this situation, $s(\tau)$ must be a regular solution. More precisely, we must have $s \equiv w$, yielding to

$$\Delta(w;\tau) \;=\; \Delta_0(w) - k_d \widehat{\Delta}_\tau(w) P(w), \quad \Rightarrow D(w) \equiv \Delta_0(w)$$

and

$$R(w,\tau) \equiv -k_d \widehat{\Delta}_\tau(w) P(w).$$

Therefore, since both $\widehat{\Delta}_\tau$ and P are continuous functions of w and τ, it is clear to see that $w(\tau)$ can be written as, $w(\tau) = w^* + g(\tau)$, where $g(\tau) \to 0$ when $\tau \to 0^+$. These arguments simply prove that there exist always $n-$regular solutions of Δ converging to the solutions of Δ_0. Clearly, in this case, the above arguments show the property (i) and the first situation in (ii).

Consider now the case $\beta \in \mathbb{Q}_+$. Under this assumption, from (4.24) it is clear that $\alpha \in \mathbb{Q}_+$, otherwise conditions (a)-(b) will not be fulfilled. Hence, bearing in mind the above observation, from (4.23) we have,

$$\begin{aligned}\widetilde{\Delta}(w;\tau) &= \tau^\alpha \Delta\left(\tau^{-\beta} w; \tau\right), \\ &= \tau^\alpha \Delta_0\left(\tau^{-\beta} w\right) - k_d \tau^{\alpha-1}\left[\tau \widehat{\Delta}_\tau\left(\tau^{-\beta} w\right)\right] P\left(\tau^{-\beta} w\right).\end{aligned} \tag{4.25}$$

According to (4.7) we have:

$$\tau\widehat{\Delta}_\tau\left(\tau^{-\beta}w\right)=\tau\sum_{j=2}^{\infty}(-1)^j\frac{\tau^{j-1}}{(j)!}\left(\tau^{-\beta}w\right)^j=\sum_{j=2}^{\infty}\frac{(-1)^j}{(j)!}\tau^{j(1-\beta)}w^j. \tag{4.26}$$

From (4.26), observe that in order to fulfill property (b), we must restrict β to satisfy the inequality $\beta\leq 1$ and, since $\beta\in\mathbb{Q}_+$, it follows that $0<\beta\leq 1$. If $\beta\in(0,1)$, then $\exists R(w,\tau)$, satisfying property (ii), however, under such circumstances, from the definition of P, we have to choose α satisfying the inequality:

$$\alpha-1-\ell\beta\geq 0, \tag{4.27}$$

where $\ell\in\{0,1,\ldots,m\}$, that is, $\ell\leq n-1$. On one hand, in the case of having $\ell<n-1$, inasmuch as $\mathrm{ord}_\tau\left(\Delta_0\left(\tau^{-\beta}w\right)\right)+\alpha\geq 0$, this will imply that $-k_d\tau^\alpha\widehat{\Delta}_\tau\left(\tau^{-\beta}w\right)P\left(\tau^{-\beta}w\right)$ is a part of $R(w,\tau)$, Moreover, in this situation $\alpha=n\beta$, and thus, $D(w)\equiv w^n$, which clearly does not fulfill the property (a). On the other hand, if $\ell=n-1$, i.e. $m=n-1$, one gets:

$$\alpha=n\beta+(1-\beta)\quad\Rightarrow\alpha>n\beta\quad\Rightarrow D(w)\equiv 0.$$

Hence, in the light of the above arguments it follows that the only possibility is to have $\beta=1$. In such a case, $\tau\widehat{\Delta}_\tau\left(\tau^{-\beta}w\right)=\widehat{\Delta}_1(w)$, implying that D will be composed by terms of $\tau^\alpha\Delta_0$ and by terms of $-k_d\tau^{\alpha-1}\widehat{\Delta}_\tau P$, having roots different from the trivial ones, showing that for the sequence (τ_j) it holds that $\tau_j s(\tau_j)$ converges to a non zero complex number, proving (i) and the second case in (ii). □

Remark 4.6.1 (Regular/singular solutions)**.** *In the proof above, it is important to point out that* 0 *is an accumulation point of* $\mathbb{R}_+$ *and the corresponding characteristic function of the closed-loop system is properly defined and is continuous at* $\tau=0$ *(i.e. the free-delay case). Based on this last observation and on the proof arguments, it is straightforward to deduce that the case when* $\tau_j s(\tau_j)\to 0$ *when* $\tau_j\to 0^+$ *for* $j\to+\infty$*, corresponds to the so-called* regular *characteristic roots seen as function of the delay parameter* τ*. For the sake of brevity, we will call such roots as* regular solutions*. Furthermore,* $\tau\equiv 0$ *will correspond to a* regular point*.*

By analogy, the case $\tau_j s(\tau_j) \to z \neq 0$ characterizes the singular *solutions. In this last situation, the singular solution will be called* regular singular solution *of* first-order *since $\tau = 0$ appears as a first-order* isolated singularity *(pole) of the corresponding solution. Furthermore, the corresponding* singular point $\tau \equiv 0$ *will be called* regular singular point of first-order.

To resume, the Proposition above simply says that when $\tau \to 0^+$, the singular solutions of the characteristic function of closed-loop system (4.1)-(4.4) *as functions of τ are singular regular solution of the first-order, and that the "isolated" singularity $\tau \equiv 0$ is a regular singular point of first-order.*

Remark 4.6.2. *It is worth mentioning, that an extension of Lemma 4.6.1 to a broader class of dynamical systems was presented and discussed in [58]. However, the arguments given there were based on the Newton Diagram Method and the Implicit Function Theorem.*

The following result will explore the existence of singular roots by considering an *auxiliary* quasi-polynomial of $\widetilde{\Delta}(w;\tau)$, where such a function is derived from (4.23), as follows

$$p_0(w) \quad := \quad \lim_{\tau \to 0^+} \widetilde{\Delta}(w;\tau). \tag{4.28}$$

Proposition 4.6.1. *Consider the characteristic function $\Delta(\cdot;\cdot)$ be given by* (4.9) *and assume that* $\deg Q \equiv \deg P + 1$. *Then, the auxiliary quasi-polynomial p_0 is given by:*

$$p_0(w) = w^{n-1}\left(w + k_d\left(1 - e^{-w}\right)p_{n-1}\right). \tag{4.29}$$

Moreover, if z_k is a non trivial characteristic root of p_0, then the characteristic root $s(\cdot) : \mathbb{R}_+ \mapsto \mathbb{C}$ of $\Delta(\cdot;\cdot)$ as a function of the delay $\tau \in \mathbb{R}_+$ can be written as:

$$s(\tau) = \frac{z_k}{\tau} + \frac{(q_{n-1} + k_p p_{n-1})z_k + k_d(1 - e^{-z_k})b_{n-2}}{k_d p_{n-1}\left[(1 - e^{-z_k}) - z_k e^{-z_k}\right]} + \mathcal{O}(\tau). \tag{4.30}$$

Remark 4.6.3. *In the light of the Remark 4.6.1, the solution $s(\cdot)$ given by* (4.30) *is a regular singular solution of first-order.*

Proof. According to Lemma 4.6.1 and Remark 4.6.1, we have that the characteristic function Δ can only have regular singular solutions of first-order. Moreover, such a solution exists if and only if $\deg Q \equiv \deg P + 1$. Since, such a condition is fulfilled by hypothesis, it follows that this solution exists and, in the light of Lemma 4.6.1,

$$s(\tau) = \frac{z_k}{\tau} + \mathcal{O}(1),$$

where z_k is a nontrivial solution of $D(w)$. Now, since $\deg Q = \deg P + 1 \Rightarrow \beta = 1 \wedge \alpha = n$, it follows that:

$$\begin{aligned} p_0(w) = (w) &= (1 + k_d p_{n-1})\, w^n - k_d p_{n-1} w^{n-1} \widehat{\Delta}_1(w) = \\ &= w^{n-1} \Big((1 + k_d p_{n-1})\, w - k_d p_{n-1} \widehat{\Delta}_1(w) \Big). \end{aligned} \tag{4.31}$$

Now, since $\widehat{\Delta}_1$ satisfies the relation $\widehat{\Delta}_1(w) = e^{-w} + w - 1$, it follows that (4.31) can be expressed as:

$$\begin{aligned} p_0(w) &= w^{n-1} \left((1 + k_d p_{n-1})\, w - k_d p_{n-1} \left(e^{-w} + w - 1 \right) \right) = \\ &= w^{n-1} \left(w + k_d \left(1 - e^{-w} \right) p_{n-1} \right). \end{aligned} \tag{4.32}$$

Finally, after substituting $s = z_k/\tau + \xi$ in Δ and after some simple algebraic (but, tedious) manipulations, we retrieve expression (4.30), concluding the proof. □

Remark 4.6.4. *Observe that according to the Remark 4.6.1, we consider $\tau = 0$ as a singular point if and only if the characteristic root $s(\tau)$ is a singular solution of first-order in $\mathbb{C}_+$. In other words, of all singular solutions are regular singular solutions of first-order.*

The following result give conditions to characterize the singular solutions of Δ.

Theorem 4.6.1. *Consider the controller C_τ given in* (4.4) *in closed-loop with system H_{yu}. Then, the following statements hold:*

i) If $\deg(Q) > \deg(P) + 1$, then all the solutions are regular and $\tau \equiv 0$ is a regular point;

ii) If $\deg(Q) = \deg(P) + 1$, then there exists singular solutions if and only if $k_d p_{n-1} + 1 < 0$. In this case, $\tau \equiv 0$ is a regular singular point of first-order.

Proof. (i) It follows straightforwardly from Lemma 4.6.1. (ii) It is a direct consequence from Lemma 4.6.1 and Proposition 4.6.1. □

Remark 4.6.5. *The leading coefficient $z_k(\tau)$ of the singular solution* (4.30) *can be computed as*

$$z_k = \mathcal{W}\left(k_d p_{n-1} e^{k_d p_{n-1}}\right) - k_d p_{n-1}. \tag{4.33}$$

Corollary 4.6.1. *The system H_{yu} in closed-loop with the delay approximation scheme $C_\tau(s)$ is said to be ill-posed if and only if the system's relative degree is equal to* 1 *and $k_d p_{n-1} + 1 < 0$.*

Condition (ii) in Theorem 4.6.1 describes the region of the parameter space (k_d, p_{n-1}), where the system posses singular solutions. Figure 4.3 illustrates such a situation.

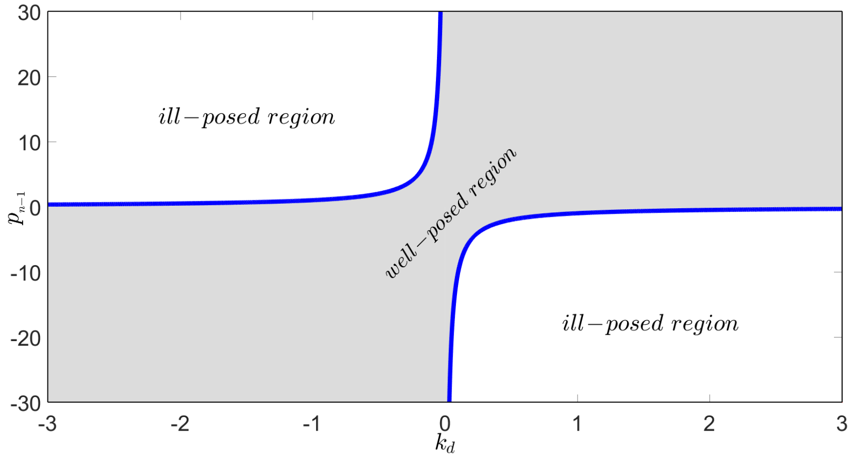


Figure 4.3: Well/ill-posed region in the (k_d, p_{n-1}) −parameter space for the closed-loop region.

4.6.1 Well-Posed Case

In the ill-posed case, conditions for the existence of singular roots for small delays were given. The following proposition gives conditions for the stabilization of system H_{yu} in the well-posed case.

Proposition 4.6.2 (Stability crossing directions)**.** *Let $\left(k_p^*, k_d^*\right) \in \mathbb{R}^2$ be fixed gains,* $\deg(Q) > \deg(P) + 1$ *and assume that $P(\boldsymbol{i}w) \neq 0$ for all $w \in \mathbb{R}$. Assume further that $(\omega^*, \tau^*) \in \Lambda$, such that $\boldsymbol{i}\omega^*$ is a simple root of $\Delta(s; \tau^*)$. Then, for $\tau > \tau^*$*

sufficiently close to τ^*, a pair of zeros of Δ will enter to the right half-plane $\mathbb{C}_+$ (or to the left half- plane $\mathbb{C}_-$) if the following inequality holds:

$$\text{sign}\left\{\Re\left[\mathcal{S}\left(\boldsymbol{i}\omega^*, \tau^*\right)\right]\right\} > 0 \ (< 0), \tag{4.34}$$

where the function $\mathcal{S} : \partial\mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ is defined as:

$$\mathcal{S}\left(s, \tau\right) := \frac{\tau Q'\left(s\right) + \left(\tau k_p + k_d\right) P'\left(s\right) + k_d\left(\tau P\left(s\right) - P'\left(s\right)\right) e^{-\tau s}}{k_d P\left(s\right)\left(1 - \left(\tau s + 1\right) e^{-\tau s}\right)},$$

where $Q'\left(s\right) := \frac{d}{ds} Q(s)$ and $P'\left(s\right) := \frac{d}{ds} P(s)$.

Proof. The proof follows straightforwardly from the implicit function theorem applied to the characteristic function $\Delta(\cdot, \cdot) : \mathbb{C} \times \mathbb{R}_+ \mapsto \mathbb{C}$ at the point $(\boldsymbol{i}\omega, \tau^*)$ under the assumption that the corresponding characteristic root is simple. More precisely, in this case, the derivative of the function s along τ writes as:

$$\left[\left.\frac{ds}{d\tau}\right|_{(\tau^*, \boldsymbol{i}\omega)}\right]^{-1} = \left.\frac{\tau Q'\left(s\right) + \left(\tau k_p + k_d\right) P'\left(s\right) + k_d\left(\tau P\left(s\right) - P'\left(s\right)\right) e^{-\tau s}}{k_d P\left(s\right)\left(1 - \left(\tau s + 1\right) e^{-\tau s}\right)}\right|_{(\tau^*, \boldsymbol{i}\omega)}. \tag{4.35}$$

Thus, if the real part of (4.35) is positive, a pair of roots of Δ will cross to the right half-plane $\mathbb{C}_+$. Furthermore, the crossing will be to the left half-plane C_-, if the real part of (4.35) is negative, which completes the proof. □

4.7 Illustrative Examples

In this section, several examples are presented in order to show the presence of singular roots. The first example is taken from the literature, where roots with singular behavior are observed. Next, an example is proposed in order to show the effect on the stability properties when the derivative approximation is applied. For the following examples, the software package QPmR have been for the numerical computations (see, for instance, [88]).

Example 4.7.1. *Unstable Implementation. Second Order System*

Let's consider the system described by the following transfer function,

$$H_{yu}(s) = \frac{-s}{s^2 - \omega_0^2}. \tag{4.36}$$

As a first step, the conventional controller $PD : -k_p - k_d s$ *in closed-loop with* (4.36) *is considered. By performing such steps, the following closed-loop characteristic function is derived:*

$$\Delta_0(s) := (1 - k_d)s^2 - k_p s - \omega_0^2. \quad (4.37)$$

In plain sight, by choosing $k_d > 1$ *and* $k_p > 0$ *the closed-loop system will be asymptotically stable. Now, if instead of choosing such an "ideal" controller, the delay-based scheme* $C_\tau : k_p + k_d[1 - \exp(-s\tau)]\tau^{-1}$ *is considered, obtaining the quasi-polynomial* f_τ *given by:*

$$\Delta(s;\tau) := s^2 - \omega_0^2 - s\left(k_p + k_d\frac{1 - e^{-s\tau}}{\tau}\right). \quad (4.38)$$

It is worth mentioning that $deg(Q) = deg(P) + 1$*, which appears to be a characteristic of systems with singular roots. In this case that the system* $H_{yu}C_\tau$ *is called* ill-posed*. Furthermore, an auxiliary quasi-polynomial is defined as,*

$$p_{\tau_0}(w) = w - 2\left(1 - e^{-w}\right).$$

The non-zero root z_k *of* p_{τ_0} *is given by the value* $z_k = 1.593624$*. Hence, the singular solution* (s_1) *of the closed-loop system is given by:*

$$s_1(\tau) = \frac{1.5936}{\tau} - 0.4142 + \mathcal{O}(\tau).$$

Which determines the leading term of the singular solution (4.38) *as shown in*

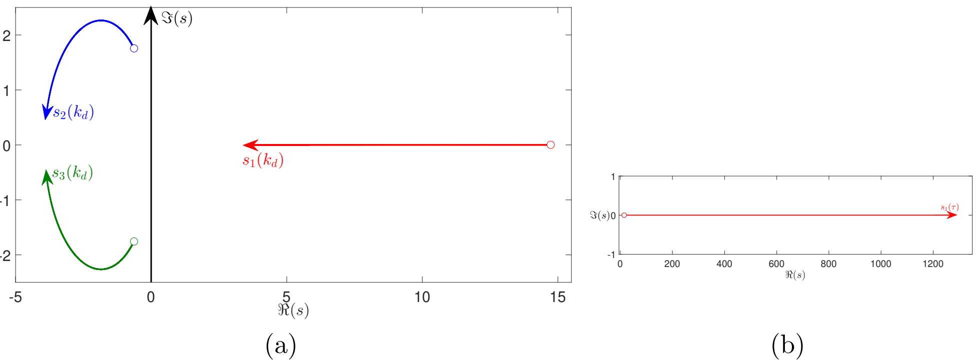


Figure 4.4: Rightmost root behavior of the closed-loop system (4.36).

Fig. 4.4. Additionally, since $k_d p_{n-1} < -1$*, as* k_d *decrease the singular solution*

will move from right to left, as can be appreciated from Fig. 4.4a. Moreover, if instead of decreasing k_d, τ is decrease, the singular solution will move from left to right, as can be observed from Fig. 4.4b.

Example 4.7.2. *Unstable Implementation. Third Order System*

In the same spirit of the previous example, confider the system described by the following transfer function,

$$H_{yu}(s) = \frac{1 - 5s^2}{s^3 + 8s^2 - 13s - 8}. \tag{4.39}$$

It is not difficult to see that the pair $(k_p, k_d) = (4, 1)$ stabilize the system in closed-loop.

Now, as in the previous example, $deg(Q) = deg(P) + 1$ and $k_d p_{n-1} = -5$, hence, in according with Lemma 4.6.1, the system $H_{yu}C_\tau$ is in the ill-posed case. For this system the auxiliary polynomial rewrites as,

$$p_{\tau_0}(w) = w - 5\left(1 - e^{-w}\right).$$

In consequence, by computing the root z_k of p_{τ_0}, the numerical value $z_k = 4.9651$ is found. Hence, by Proposition 4.6.1, the singular solution (s_1) of the closed-loop system is given by:

$$s_1(\tau) = \frac{4.9651}{\tau} + 12.4337 + \mathcal{O}(\tau). \tag{4.40}$$

Thus, according to (4.40), choosing $\tau_0 = \frac{1}{5}$, the singular root s_1 will be located approximately to $s_1(\tau_0) \approx 36.8255$, which is very close to its real value $s_1(\tau_0) = 37.3332$. Observe that if the delay decrease its value to $\tau_0 = \frac{1}{80}$, the singular root s_1 will be located approximately to $s_1(\tau_0) \approx 409.208$, which is very close to its real value $s_1(\tau_0) = 409.6421$. The complete behavior of the rightmost solutions can be appreciated from the figure 4.5.

Example 4.7.3. *Stabilizing Effect of C_τ*

As a final example, it is shown that the use of C_τ may be beneficial for some dynamical systems where the standard PD-controller C_0 does not stabilize the system. In this regard, let's illustrate the advantage of the controller C_τ over the

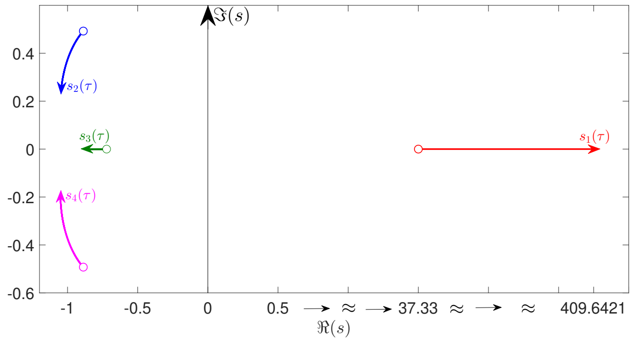


Figure 4.5: Rightmost root behavior of the closed-loop system (4.39).

static controller C_0, through the stabilizing effect on the following unstable open-loop system:

$$H_{yu}(s) = \frac{1}{s^3 - s^2 + 4s - 31}.$$

In fact, system H_{yu} has one unstable root at $s_1 = 3$. It is evident to see, that such a system cannot be stabilized by using a conventional PD-controller. Thus, our aim will be to analyze if this plant, can be handle by the delay-based scheme. The closed-loop characteristic function is described by the quasi-polynomial:

$$\Delta(s;\tau) := \left[s^3 - s^2 + 4s + \left(k_p + \frac{k_d}{\tau} - 31\right)\right] - \frac{k_d}{\tau}e^{-s\tau}. \tag{4.41}$$

Since the main concern is the stability analysis of the closed-loop system, the delay interval is fixed to some compact set $\mathcal{D} = [0, \tau_u]$ with $\tau_u > 0$. For the sake of convenience let us fixed the parameters $(k_p, k_d) = (32, -3)$. First, the roots the system $H_{yu}C_0$ are investigated, who stability is determined by the characteristic polynomial $f_0(s)$ defined by $f(s,\tau) \to 0^+$:

$$\Delta_0(s) = s^3 - s^2 + s + 1.$$

Hence, the rightmost roots of the delay-free system are given by $s_{2,3} = 0.77184 \pm \boldsymbol{i}1.11514$. Such a behavior can be seen in figure 4.6. From this, it is concluded that the unstable characteristic roots cross the imaginary axis from the right-hand plane to the left-hand plane as τ increases from τ^l to τ^l, which satisfy the equation.

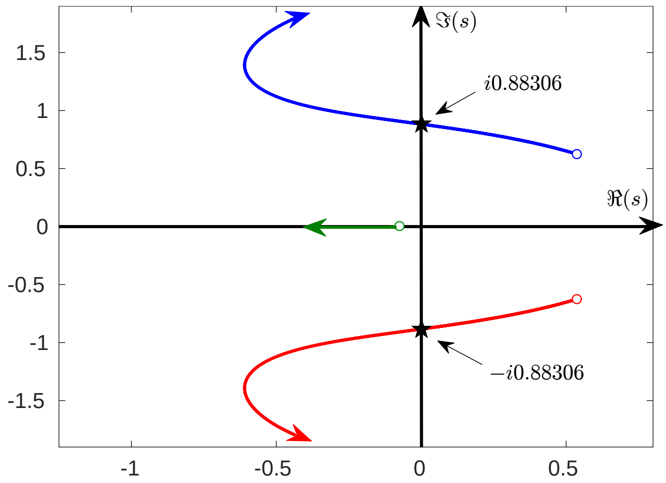


Figure 4.6: Root Crossing of the characteristic equation (4.41).

4.7.1 Numerical Evaluation of the Closed-Loop

The presence of singular roots of the control loop has been studied analytically in the previous section, by means of the computation of the root with singular behavior depending on the delay. In what follows the goal is to implement a numerical verification of such analysis. For this purpose, the closed-loop system $H_{yu}C_\tau$ is implemented using Matlab to describe the response of the system.

Example 4.7.4. *Well-Posed Second Order System*

The root behavior for small delays was described by Lemma 4.6.1, and to complement this analysis, in this example the following system is considered:

$$H_{yu}(s) = \frac{s + 0.5}{s^2 - 3s - 1}, \tag{4.42}$$

with controller given by $C_\tau := k_p + k_d(1 - \exp(-s\tau))\tau^{-1}$*. It is clearly seen that* H *satisfy* $deg(Q) = deg(P) + 1$*, therefore the first condition in Theorem 4.6.1 is satisfied but the condition* $k_d p_1 < 0$ *is not, in such a way that the system is well-posed. Now, with the choice of gains* $(k_p, k_d) = (3, 3.5)$*, the step response of the system* $H_{yu}C_\tau$ *is shown in figure 4.7. Following these observations, it can be assured that the system is well-posed when* C_τ *is applied.*

Example 4.7.5. *Stabilization of a Third Order System*

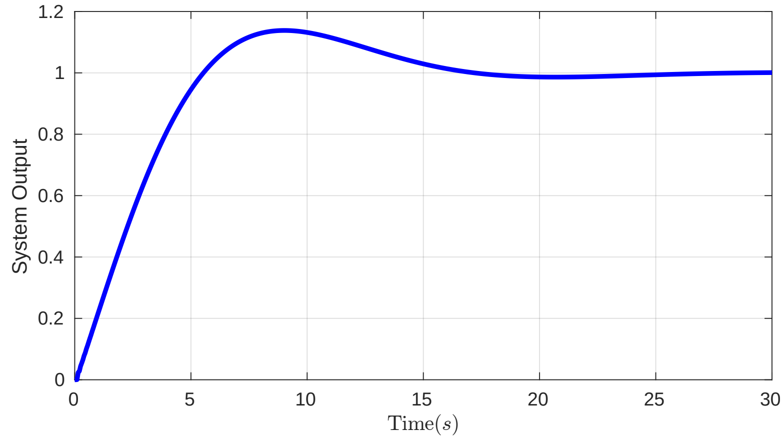


Figure 4.7: Output response of the closed-loop system $H_{yu}C_\tau$ (4.42).

This example highlights the destabilizing effect of the time-delay along with the choice of gains (k_p, k_d) considering the controller C_τ. The system to consider is given by

$$H_{yu}(s) = \frac{s+1}{s^3 - 2}. \tag{4.43}$$

It's easy to see that the controller C_0 can stabilize the system with gains (k_p, k_d) satisfying $k_p > 2$ and $k_d > \left(\sqrt{k_p^2 - 4k_p - 8} - k_p\right)/2$. With the choice of gains $(k_p, k_d) = (3, 2)$ for the controller C_0, the step response of the system H_{yu} is depicted in figure 4.8. Now, the controller C_τ is considered instead. The gains are set

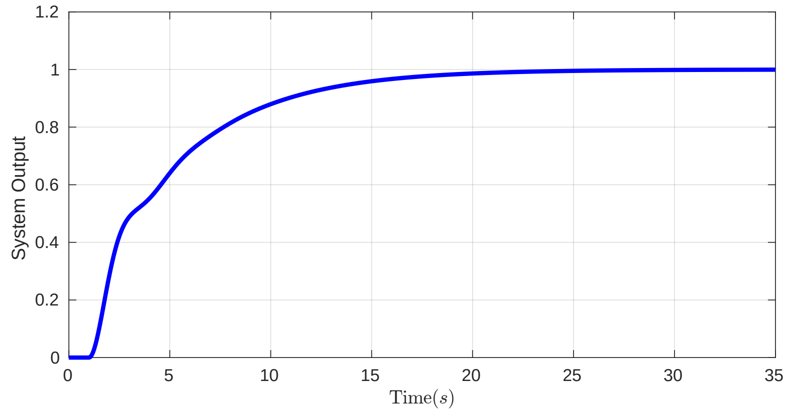


Figure 4.8: Output response of the closed-loop system $H_{yu}C_0$ (4.43).

as $(k_p, k_d) = (3.5, 1.8)$ and the delay $\tau = 0.65$, slightly different to C_0 a unstable *system $H_{yu}C_\tau$ is obtained as shown in figure 4.9. The reason for this behavior is*

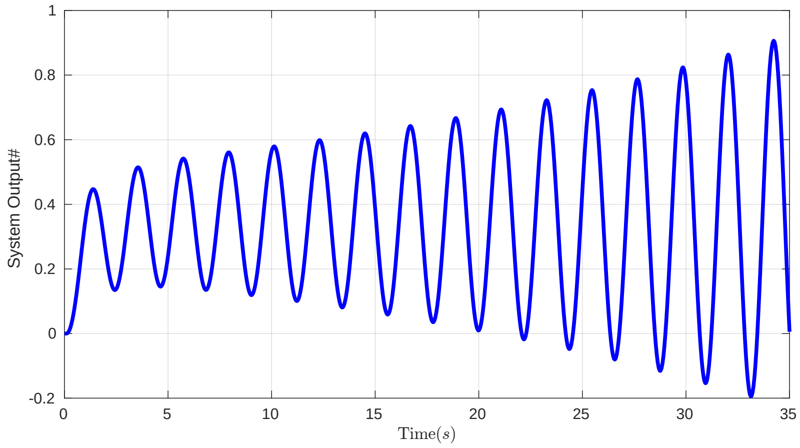


Figure 4.9: Output response of the closed-loop system $H_{yu}C_\tau$ (4.43).

the presence of roots in the right half-plane induced by the delay but with regular behavior.

4.8 Chapter Summary

The main subject of this chapter is the PD control scheme with delay approximation of the derivative action. The focus is on the study of some subtitles that arise when considering small delay values in the ill-posed case.

Concerning the asymptotic analysis for singular roots, an emphasis was placed on the approximation of the derivative action for small delay values, managing to characterize in a simple way the behavior of the roots. The problem was solved by calculating the series of Laurent roots that intersect at infinity. The characterization presented in this chapter gives a deep understanding of the Ill-posed interconnection scheme $H_{yu}C_\tau$ using the approximation derivative by an explicit representation of the singular solution. Resulting in a simple condition for the presence of an abrupt change in the qualitative behavior for the transition $\tau = 0$ to $\tau \to 0_+$.

Perspectives and Conclusions

This thesis introduces a method for the asymptotic stability analysis of a class of LTI-SISO subject to time-delay. The focus is on some particular cases of critical roots, mainly on the characterization of multiple roots in Chapter 2, then, in Chapter 3, the proposed methodology is extended to two delay parameters. However, there still exist some open problems, in particular, those concerning the characterization of the root behavior under multivariable variations, as well as the characterization of the singular solutions (discussed in Chapter 4).
As an extension, the asymptotic behavior of critical roots of linear time-delay systems with delay-dependent coefficients is also considered using the proposed method.

Quasi-polynomials with Delay-Dependent Coefficients

For some particular systems, the delay may also appear in the quasi-polynomial coefficients. Such a case can be found in different disciplines, for instance, in the analysis of the rate of convergence of some control systems [44, 79], when modeling some biological systems, like population dynamics, immune dynamics in leukemia, competition in a chain of chemostats [46, 26, 74, 62]. The stability of this type of system can be analyzed by considering the following quasi-polynomial:

$$f(s,\tau) = p_0(s,\tau) + p_1(s,\tau)e^{-s\tau}. \tag{4.44}$$

It is assumed that p_0, p_1 are polynomials in λ and continuous functions in the τ, for some particular delay interval of interest $\tau \in [0, \tau^u)$. In this vein, two different kind of characteristic functions $f(\lambda,\tau)$ are considered. This first kind, in which all the roots are continuous. Hence, the stability analysis can be achieved by the crossing of the imaginary roots $\boldsymbol{i}\omega$ as τ approach a critical delay value τ^*. However, for the second kind, some roots present *singular behavior*, namely, roots that go infinity as $\tau \to 0^+$, as explained in Chapter 4 for the case of the derivative approximation. The following example shows a system where the approximation of the derivative is used, in such a way that a quasi-polynomial with delay-dependent coefficients is obtained. Furthermore, the ill-posed situation may be present for small delay values in a different scheme than the one in Chapter 4.

Example 4.8.1. *Gradient Play Dynamics, Ill-Posed/Well-Posed*

In order to show an ill-posed situation for systems in a more general configuration, let's consider the approximation of the derivative action on a continuous Gradient Plat Dynamics model [84]. In [84], via a geometric approach, the authors analyze the effect of the delay on the stability of the system in which some subtleties need to be considered. Let s_i *be the eigenvalues of the linearization without derivative action, then the characteristic function associated with each eigenvalue with time delay is given by*

$$f(s,\tau) = \left[s - s_i\left(1 + \frac{\gamma}{\tau}\right)\right] + s_i\frac{\gamma}{\tau}e^{-s\tau}. \tag{4.45}$$

When $\tau = 0$*, stability conditions can be found by tuning* $\mid \lambda_i \mid$ *and* γ*. Furthermore, for a small delay (*$\tau \to 0^+$*), the above quasi-polynomial may possess roots at infinity. Nonetheless, by tuning the parameters to* $|\lambda_i| \cdot \gamma < 1$*, there may be situations where all the roots possess a negative real part for small delay values.*
Applying the change of variable given in 4.6.1 from Chapter 4, for $\beta_0 = 0$ *and* $\nu_0 = 1$*, define the associated quasi-polynomial* $f_0(s,\tau) = \tau^{-\nu}\tilde{f}(\tau^{\beta_0}w,\tau)$ *given by:*

$$\tau^{-1}\tilde{f}(w,\tau) = w - s_i - \frac{s_i\gamma}{\tau}\left(1 - e^{-w\tau}\right).$$

Thus, from the quasi-polynomial $\tilde{f}$*, the auxiliary* $p_{f_0}(w) = \lim_{\tau\to 0} f$ *is given by*

$$p_{\tau_0}(w) = w(1 - s_i\gamma) - s_i,$$

which coincides what was discussed in Chapter 4.

In the light of the analysis given in the previous example, it is clear to see that some subtleties arise when considering the influence of the delay in the parameters in a border class of systems. Hence it will interesting to extend the results presented in this thesis to such kinds of systems. A conjecture to tackle such problems is to exploit the structure of the singular solution by means of computing the Laurent series and to analyze its behavior for small delay values around a point at infinity.

For future work, it is proposed to extend the work done during the thesis work to quasi-polynomials with delay-dependent coefficients in the ill-posed situation, and to a larger class of equation $f(\lambda, \tau) = 0$, by considering a wider class of meromorphic coefficients. In particular, a particular form of delay-dependent coefficients with denominator given by a polynomial in τ:

$$P_i(\lambda, \tau) = \frac{p_i(\lambda, \tau)}{q_i(\tau)}.$$

If $\tau = \tau^* \geq 0$ is a zero of q_i such that $p_i(s, \tau^*) \neq 0$, it is posible to consider poles given by real roots of the polynomial $q_i(\tau)$ and analyze the behavior when $\tau \to \tau^*$. Such situation will be presented in future work.

Multiple Variable Perturbation

The proposed approach to the asymptotic analysis of multiple roots presented in Chapter 2 was extended to the case of two parameters in Chapter 3, preserving the advantages of the methodology and covering the challenges of generalization. The solution of the equation $f = 0$ when f is given by a polynomial with several parameters has been studied in the open literature. In a broader sense for higher dimension is solved by means of the so-called *Resolution of Singularities* in Algebraic Geometry, by means of blowing-ups in $\mathbb{C}^n$. The approach taken in this thesis can be fixed into this scheme by describing the blowing of the substitutions $x_i = u_i$ and $x_2 = u_1 u_2$ that is described in a formal way.

1. extend the Newton diagram procedure to Weierstrass polynomial of several variables;

2. obtain conditions that allow obtaining Puiseux series solutions

$$s(\boldsymbol{\tau}) = c(\tau_2^{1/d}, \ldots, \tau_n^{1/d})\tau_1^{\beta} + o(\tau_1^{1/d}, \ldots, \tau_n^{1/d}),$$

where $\beta = \alpha/d$ and $\alpha \in \mathbb{N}$;

3. give conditions on $f(s, \tau)$ which describes the splitting properties of its solutions $s(\tau_1, \tau_2)$.

Scientific Contributions

During the development of the thesis, the following results were obtained.

International Conferences:

- **"Some Remarks on the Asymptotic Behavior for Quasipolynomials with Two Delays"**. *Martinez-Gonzalez, A., Mendez-Barrios, C.F., Niculescu, S.I., & Chen.* 14th IFAC Workshop on Time Delay Systems TDS, 2018.

- **"Insights in Characterizing Asymptotic Behavior for Quasi-polynomials with Two Delays"**. *Martinez-Gonzalez, A., Mendez-Barrios, C.F., Niculescu, S.I., & Chen, J. 2018.* 23rd International Symposium on Mathematical Theory of Networks and Systems.

- **"Insights on Asymptotic Behavior of Characteristic Roots of Quasi-Polynomials with Delay-Dependent Coefficients in Some Ill-Posed Cases"**. *Martinez-Gonzalez, A., Mendez-Barrios, C.-F, Niculescu, S.-I., & Chen, Jie.* 15th IFAC Workshop on Time Delay Systems 2019.

- **"Some Remarks on the Regular Splitting of Quasi-Polynomials with Two Delays. Characterization of Double Roots in Degenerate Cases"**. *Martinez-Gonzalez, A., Mendez-Barrios, C.F., & Niculescu, S.I.* IFAC World Congress 2020.

Journal:

- **"Weierstrass Approach to Asymptotic Behavior Characterization of Critical Imaginary Roots for Retarded Differential Equations"**. *Martinez-Gonzalez, A., Mendez-Barrios, C., Niculescu, S., Chen, J., & Felix, L.*. SIAM Journal on Control and Optimization.

- **"Characterizing Some Ill-Posed Problems in PD-Control"**. *César-Fernando Méndez-Barrios, Silviu-Iulian Niculescu, Alejandro Martínez-González, Adrián Ramírez* International Journal of Robust and Nonlinear Control (*Submitted*).

Book Chapter:

- **"Asymptotic Analysis of Multiple Characteristics Roots for Quasi-polynomials of Retarded-Type"**. *Martinez-Gonzalez, A., Niculescu, S.-I., Chen, J., Mendez-Barrios, C.-F, Romero, J.G., & G., Mejia-Rodriguez.* Delays and Interconnections: Methodology, Algorithms and Applications. Advances in Delays and Dynamics, Springer, 2019.

In summary, the chapters are related and set forth what is dealt with in the published articles as follows. Publication 1 and 2, deal with the computation of the Weierstrass polynomial as shown in Chapter 2 for the numerical and analytical perspectives respectively. Publications 3, 4, and 6 are the result of the study of the asymptotic behavior of multiple roots subject to variations of two delays Chapter 3. For small delay values, Section 4.8 considers the case of delay-dependent coefficients, and the description of the singular roots in scheme $H_{yu}C_{\tau}$ is expected to be published in 7.

Conclusions

Accordingly, to the previous discussion, the central question to deal with in this thesis is as follows.

Question 4.8.1. *Given a singular point of the solution $s(\tau)$ of the characteristic equation. Are the geometric properties of multiple roots completely characterized?. In the case of ill-posed systems, can the non-bounded root be represented using Laurent expansion?*

As it was shown in the previous chapters, the answer turns out to be yes. Different approaches to this problem can be found in the literature which suggests that is the case.

In particular, the analysis of the asymptotic behavior of multiple imaginary roots for quasi-polynomials is deeply studied. The thesis work is based on the Weierstrass Preparation Theorem, which allows the analysis of the local behavior of a given critical solution. Some algebraic properties, CRS, RS, and NRS, have been presented to characterize the branch structure for all critical solutions.

In the extension to multiple parameters, some issues concerning the asymptotic behavior of multiple critical roots for quasi-polynomials with two delays are considered. The proposed approach is based on an iterative Newton diagram method which can be effectively applied to find the leading terms of power series solutions expressed as a generalized Puiseux series.

Emphasis was placed on the subtitles that arise when considering small delay values when the approximation of the derivative is considered. The discussion was made through the asymptotic analysis of the singular root giving a deep understanding of the interconnection scheme $H_{yu}C_{\tau}$. It is also of importance to characterize this phenomenon in more general cases of quasi-polynomials with delay-dependent coefficients.

Appendix A

Well-posed Linear Systems

Roughly speaking, a system is well-posed if on any time interval $[\tau, t] := \Omega$, for any initial state x_0 in the state space and any input function u in a specified space of functions, it has a unique state trajectory x and a unique output function y, both defined on Ω. Moreover, y must belong to a specified space of functions, and both $x(t)$ and y must depend continuously on $x(\tau)$ and on u (see, [87] for a survey). This concept is general and can be made precise for many classes of non-linear and/or time-varying systems. However, most attention in the literature has been devoted to the simplest particular case, namely, linear and time-invariant systems, because here we have strong tools to develop the theory.

For finite-dimensional Control Systems Well-posedness is devoted to systems described by equations of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A.1}$$

$$y(t) = Cx(t) + Du(t), \tag{A.2}$$

where u is the input signal, x is the state, y is the output signal and A, B, C, D are matrices. Generally such systems are considered to evolve over the time interval $[0, \infty)$.

Generally speaking the main idea is that is that the system Σ is fully described by

$$\begin{bmatrix} x(t) \\ \mathbf{P}_{[t,\tau]} y \end{bmatrix} = \Sigma(t, \tau) \begin{bmatrix} x(\tau) \\ \mathbf{P}_{[t,\tau]} u \end{bmatrix},$$

where x is the state trajectory of the system, u, y are the inputs and outputs signals and $\mathbf{P}$ is the projection operator. Then, the operators $\Sigma(t, \tau)$ are continuous (bounded), the system Σ is called *well-posed.*

Appendix B

Stability Test Methods

A small introduction to the main methods for the stability analysis is given. Since the location of the zero set of the quasi-polynomial determines the stability of the time-delay system, the main methods presented here are in the frequency domain approach, with associated characteristic function given by the following form

$$f(s, \tau) = P(s) + Q(s)e^{-s\tau}.$$

Before continuing with the description of the main stability analysis methods, an essential result for the asymptotic analysis is given.
The following theorem (see, for instance, [4, 81]) is the basis for the Rouché's Theorem. Let Z and P denote the number of zeros and poles (counting multiplicities) of $f(z)$ contained in a simple closed contour C (without self-intersections).

Theorem B.0.1. *[81] Let $f(z)$ be a meromorphic function inside and on $\mathcal{C}$ then*

$$\frac{1}{2\pi \boldsymbol{i}} \oint_{\mathcal{C}} \frac{f'(z)}{f(z)} dz = Z - P.$$

The following theorem is often used to simplify the problem of locating zeros, hence it is of vital importance to our approach to stability analysis.

Theorem B.0.2. *Rouché's Theorem [13] Let $f(z)$ and $g(z)$ be two functions which are analytic inside and on a simple closed contour C in the complex plane. If*

$$|g(z)| < |f(z)|$$

for any z on C, then $f(z)$ and $f(z) + g(z)$ have the same number (multiplicities included) of zeros inside C.

B.1 The D-Decomposition Method

The computation of characteristic roots of time-delay systems represent challenge due to infinite number. Due to this difficulty it is desirable to determine system stability without the exact knowledge of the complete set of roots. In essence, the method due to [69], divides the parameter space into regions by hypersurfaces, and the points of which correspond to quasi-polynomials having at least one zero on the imaginary axis. Thus, in a given region the number of roots with positive real part is constant, for all parameters.

Since the characteristic roots depend on τ continuously [48], the approach becomes a convenient and powerful tool for stability analysis. In particular, when the parameter in the D-decomposition is the delay parameter, it is known as the τ-decomposition (subdivision) method.

This method can be described as follows. First, the set of critical delay values are identified, i.e., associated to the roots on the imaginary axis, satisfying

$$\tau_1 < \tau_2 < \cdots < \tau_n$$

Next, the root crossing direction analysis. In order to analyze how the number of roots with positive real parts changes when a partition is crossed, the root tendency is computed which determine the direction of the crossing from its algebraic sign. In other words, one has to determine the number of characteristic roots that will move toward $\mathbb{C}_+$ through the point $\boldsymbol{j}\omega$ as τ is varied around τ^*.

Notably, if there is not critical points, it is easy to conclude that the stability properties for the case $\tau = 0$ are preserve for all positive delay values, and we say that the stability is delay-independent. Then for sufficiently close to τ^*, $\boldsymbol{i}\omega$ will enter the right-hand (left-hand) side if

$$\Re \left\{ \frac{\frac{\partial f}{\partial \tau}}{\frac{\partial f}{\partial s}} \right\} \Bigg|_{(\tau^*, \boldsymbol{i}\omega)} > 0 \; (< 0). \tag{B.1}$$

The equation above gives a simple criterion for the crossing of simple root. In the next section the case of multiple will be presented, so the method cannot be applied and (B.1) does not hold.

B.2 Critical Delay Values and Imaginary Roots

It is well known tha an alternative method in determining the stability is to analyze the asymptotic behavior of the critical zeros on the imaginary axis. Specifically, at each critical delay value corresponding to a critical zero of the characteristic quasipolynomial, we may seek to determine whether the zero may traverse from one half plane into another; for example, the system will become unstable if a critical zero enters the open right half plane, and otherwise will remain stable if all the critical zeros remain in the left half plane.

By solving a generalized eigenvalue problem the corresponding critical delay values for the imaginary roots can be computed by following the results (presented in [24]). The following lemma was obtainen in [22].

Lemma B.2.1. *The characteristic quasi-polynomial $f(s, e^{-s\tau})$ has a critical zero on the imaginary axis if and only if the following conditions are satisfied:*

(i) $\sigma(V, U) \cap \partial\mathbb{D} \neq 0$*;*

(ii) for some $z_i = e^{-\boldsymbol{i}\theta_i} \in \sigma(V, U) \cap \partial\mathbb{D}$*, where* $\theta_i \in [0, 2\pi]$

$$\sigma\left(\sum_{k=0}^{q} A_k z_i^k\right) \cap \boldsymbol{j}\mathbb{R}_+ \neq 0.$$

The imaginary number $\boldsymbol{i}\omega_i \in \sigma(\sum_k A_k z_i^k)$*, wher* $\omega_i > 0$*, is a critical zero. The set*

$$\mathcal{T}(\omega_i) = \left\{ \frac{\theta_i + 2\pi l}{\omega_i} > 0, \quad l = 1, 2, 3, \cdots \right\},$$

contain all the critical delay values corresponding to the critical zero $\boldsymbol{i}\omega_i$*.*

The matrices needed for the above lemma are defined as

$$U = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \ddots & & & & \\ 0 & O & \cdots & I & 0 \\ 0 & O & \cdots & 0 & B_{2q} \end{bmatrix}, V = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \ddots & & & & \\ 0 & O & 0 & \cdots & I \\ -B_0 & -B_1 & -B_2 & \cdots & -B_{2q-1} \end{bmatrix},$$

where $B_m \in \mathbb{C}^{n^2 \times n^2}$ are given by

$$B_{q-m} = I \otimes A_m^T, \; B_q = A_0 \oplus A_0^T, \; B_{q+m} = A_m \otimes I.$$

B.3 The Pontryagin Criterion

In [75], fundamental results on the properties of the spectrum of quasi-polynomials. The characteristic function may be written in the form of quasi-polynomial without explicit dependence on the time-delay as follows

$$P(s, e^s) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} e^{js} \tag{B.2}$$

Before introducing the method, let's introduce some terminology. Let g and h two real functions of real variable, denoting the real and imaginary part when $s = \boldsymbol{i}\omega$ is consider on P.

Theorem B.3.1. *If the quasi-polynomial Equation* (B.2) *lie on the left hand side of the imaginary axis, then all the zeros of the functions g and h are real, simple and alternating, and*

$$\dot{h}(\omega)g(\omega) - \dot{g}(\omega)h(\omega) > 0, \tag{B.3}$$

for each ω. Moreover, if no zeros with positive real part of (B.2) *are present, it is sufficient that one of the following conditions be satisfied:*

1. *All the zeros of the function $g(\omega)$ and $h(\omega)$ are real an alternate and the inequality* (B.3) *is satisfied for at least one value of ω;*
2. *All the zeros of the function $g(\omega)$ are real and for each of these zeros $\omega = \omega_0$, condition* (B.3) *is satisfied, i.e. $\dot{g}(\omega_0)h(\omega_0)$;*
3. *All the zeros of the function $h(\omega)$ are real and for each of these zeros the inequality* (B.3) *is satisfied,i.e. $\dot{h}(\omega_0)g(\omega_0)$.*

B.3.1 Argument Principle Methods

This kind of method ca be understood as extensions of Nyquist or Michailov criteria. The main goal is to obtain conditions such that reduce the number of unstable characteristic root until taking them to a null number.

In order to obtain a condition for the absence in the characteristic quasi-polynomial $f(s, \tau)$ of roots with positive real parts, the argument principle, resumed by

$$\frac{1}{2\pi\boldsymbol{i}} \oint \frac{f\prime(z)}{f(z)} dz = Z - P$$

can be applied. The method is applied to quasi-polynomial

$$f(s,\tau) = p_0(s) + p_1(s)e^{-s\tau},$$

with polynomials $p_0(s)$ and $p_1(s)$ of degree n and $n-1$ respectively. Now suppose that f can be written as

$$f(s,\tau) = u(s)$$

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