Delayed Controllers Design for Dynamical Systems

Thesis Report #1
Student: José Enrique Hernández Díez
Thesis Director: César Fernando Méndez Barrios
C.I.E.P., Facultad de Ingeniería, Universidad Autónoma de San Luis Potosí

I. INTRODUCTION

In mechanical, electrical and electronics engineering, control theory is one of the most important subject with suitable applications in these fields. The automation of industrial processes has established the importance of control theory. Low-order controllers are one of the most widely applied strategies to controlled industrial processes (see, e.g., [1], [2], [3]). This “popularity” can be attributed to their particular distinct features: simplicity and ease of implementation.

Among low-order controllers, those of PID-type are known to be able to cope with uncertainties, disturbances, elimination of steady-state errors and transient response improvement (see, for instance, [4]-[5]). However, as reported in [4], [6], the main drawbacks of PID controllers lies in the tuning of the derivative term, which may amplify high-frequency measurement noise. Moreover, as mentioned in [1], [3], [7], the above arguments advise to avoid the derivative action in most applications. In order to circumvent the above mentioned problem, we can notice that the Euler approximation of the derivative,

\[ y'(t) \approx \frac{y(t) - y(t - \varepsilon)}{\varepsilon}, \]

for small \( \varepsilon > 0 \), suggests to replace the derivative action by using delays [8], [9].

However, it is well known that adverse effects as oscillations, instability and bandwidth sensitivity, among others, are the consequence of the presence of delay in the control loop (see, for instance, [10], [11]). Also, it is worth mentioning that there exist some situations when the delay may improve the stability of the system as explained in the classical example [12], [13], where an oscillator is controlled by one delay “block”: (gain, delay), with positive gains and small delay values (a detailed analysis of such an approach can be found in [14]).

In this report we study the closed-loop stability analysis of two different delayed control schemes for an electrical and a mechanical applications. First, we propose the use of a \( P - \delta \) controller for the stabilization of a buck DC/DC converter. Second, we focus on the stabilization problem of the so-called Furuta Pendulum (see, Fig. 8) on its unstable equilibrium point at the upright position considering a time-delay in the state feedback loop.

On one hand, for the buck DC/DC converter, the controller consists of a low-order controller with two blocks: proportional + proportional-delayed. The control law for this type of scheme is given as:

\[ u(t) := k_p e(t) + k_\delta e(t - \tau), \]

where \( k_p, k_\delta \) and \( \tau \) are real parameters and \( e(t) \) is the error signal of the control scheme. On the other hand, for the delayed Furuta Pendulum, we propose the use of some standard Linear Quadratic Regulation (LQR) controller. In order to consider this scenario in the control scheme design, we use the results shown in [15] for the inverted pendulum and cart system. This shows a simple method on how to compute the critical delay value in the state feedback loop in which the closed-loop system loses stability. Moreover, we explore the behavior of the stability conditions by considering an auxiliary pair of gains for the position regulation of both angles of the Furuta pendulum. It is worth noticing that even in the simple case of the inverted pendulum, the presence of delays in the input may induce some unexpected properties as, for instance a triple root at the origin (see, for instance [16] and the references therein).

The remaining part of this report is organized as follows: Section III concerns a deep analysis of a control scheme using a \( P - \delta \) controller for the buck DC/DC converter. Section IV shows the design of a state feedback controller considering a time-delay for the Furuta pendulum. Section V discusses some concluding remarks in both analysis. Finally, Section VI shows the activities schedule for future work.

II. OBJECTIVES

A. General Objective

Analyze the use of delayed controllers and its applications to dynamical systems. Considering linear, nonlinear, delayed, time variant or invariant systems. Using as a reference the following design objectives:

- Closed-loop system stability.
- Transient response manipulation.
- Robustness under external disturbances.
- Open loop system uncertainty, whether it is parametric uncertainty or due to non modeled dynamics.
- Controllers fragility.

B. Particular Objectives

1) Proposing a variety of delayed controllers settings based on PID controllers for LTI systems also considering time delays inherent to the systems nature.

2) Developing a methodology that allows to find at least necessary conditions for the parameters of delayed
controllers (gains or delays) in order to the closed-loop system is stable.
3) Implementing delayed control schemes based on Lyapunov functions to nonlinear systems.
4) Realizing the necessary analysis in order to cover partially or totally the design objectives shown above. Particularly, developing a method to characterize the stability regions in the parameters space (gains or delays).
5) Implementing analytically and experimentally the use of delayed controllers to real systems in robotics, electrical machines or power electronics.

III. POWER ELECTRONICS APPLICATION: CONTROLLED BUCK DC/DC CONVERTER WITH A PROPORTIONAL DELAYED CONTROLLER

The generation, conversion and transmission of electrical energy has raised awareness of the relevance of power electronics in today’s applications; among the most popular applications we may consider are those related to renewable energies. This fact has established power electronics as an important subject in electrical and electronics engineering. The basic topologies in electrical conversion systems concerns to AC/DC, AC/AC, DC/AC and DC/DC. This report considers a buck converter, which due to its remarkable efficiency and simplicity is one of the most popular DC/DC converters in power electronics.

We propose the use of a \( P - \delta \) controller for the stabilization of a buck DC/DC converter. The proposed approach in this work includes a deep analysis of the closed-loop characteristic equation, which considers only a delayed term due to the controller nature. This involves problems such as stability, \( \sigma \)-stability and controller fragility, which are studied individually in the following sections.

The organization of this sections is given as follows: Section III-A discusses the modelling of a buck DC/DC converter. Section III-B is the most important contribution of this work and it concerns to the stability criterion of the closed-loop system in two manners: considering any possible delay value (independent-delay stability) and a fixed delay value. Section III-C studies the \( \sigma \)-stability and fragility problems. Section III-D shows some illustrative results for the application of the methods developed in this work and some simulation results.

A. Preliminary Results

Figure 8 depicts the classical topology of a Buck DC/DC converter, where \( v_s \) and \( v_o \) are the supply and output voltage, respectively. This configuration contains four basic elements: inductor (\( L \)), capacitor (\( C \)), diode (\( D \)) and a controlled switch (\( Q \)). Furthermore, a resistive load \( R \) is assumed. Then, considering a fixed DC voltage supply \( v_s \), the main idea is to use the switching pattern of \( D \) in order to adjust the output voltage \( v_o \). The most widely used switching technique is the PWM scheme, which consists of creating a switching pattern of \( D \) at a fixed frequency \( f \) with an activation period \( t_{on} \) such that the duty cycle of the PWM is given as \( U := f \cdot t_{on} \).

Motivated by the remarks presented in [17], a linear control formulation can be provided by assuming that all variables have a constant value and a fluctuating part, i.e.,

\[
\begin{align*}
v_s(t) &= V_s + \tilde{v}_s(t), \\
v_o(t) &= V_o + \tilde{v}_o(t), \\
i_o(t) &= i_o + \tilde{i}_o(t), \\
u(t) &= U + \tilde{u}(t).
\end{align*}
\]

Since a resistive load is assumed, then from a control theory perspective, the problem can be formulated as the task to reduce the variations at the output voltage \( v_o(t) \) despite of possible disturbances in the supply voltage \( \tilde{v}_s(t) \) and variations in the load by adding a correction factor \( \tilde{u}(t) \) to the nominal duty cycle \( U \).

The dynamic model of the buck DC/DC converter is derived assuming that the system runs in a continuous-conduction mode (CCM). The dynamic model is obtained by defining two operation modes for the switching device \( Q \): ON(\( \mu = 1 \)) and OFF(\( \mu = 0 \)). Applying Kirchoff’s law to both equivalent circuits, a switched model is derived. Consequently, by considering a PWM switching pattern, an averaged state-space model can be obtained (see, for instance, [18]), where the averaged states \( [x_1, x_2] := [i_L, v_o] \) are defined as:

\[
x_1 := \frac{1}{T} \int_{-T}^{t} i_L(h)dh, \quad \text{and} \quad x_2 := \frac{1}{T} \int_{-T}^{t} v_o(h)dh.
\]

Integrating the switch state \( \mu \) over the commutation period \( T \), a new control variable \( u := \frac{1}{T} \int_{-T}^{t} \mu(h)dh \) is defined and it represents the duty cycle. Then, the averaged model of the buck DC/DC converter considering variations in the supply voltage \( v_s \) is given as:

\[
\begin{align*}
\dot{x}_1 &= -\frac{x_2}{L} + \frac{V_s + \tilde{v}_s}{L} u, \\
\dot{x}_2 &= -\frac{x_1}{C} - \frac{x_2}{RC}.
\end{align*}
\]

Remark 1: It is worth mentioning that this averaged model can describe the nature of the system only if the commutation frequency \( f \) is sufficiently large.

Now, the relations in the converter for the mean values \( (v_s, v_o, i_o, U) \) can be derived from (6) by setting the derivatives equal to zero, which leads to:

\[
\begin{align*}
i_L &= \frac{V_o}{R}, \\
v_o &= UV_s.
\end{align*}
\]
Finally, taking a linear approximation from (6) around the nominal conditions, two transfer functions with respect to the variations in the output voltage \( \tilde{V}_o \) can be defined as:

\[
G_1(s) := \frac{\tilde{V}_o(s)}{u(s)} = \frac{V_s}{s^2 + \frac{1}{RC} s + \frac{1}{LRC}}, \tag{8}
\]

\[
G_2(s) := \frac{\tilde{V}_o(s)}{\tilde{V}_o(s)} = \frac{U}{s^2 + \frac{1}{RC} s + \frac{1}{LRC}}. \tag{9}
\]

A linear control approach is considered for ensuring stability in the closed-loop system. More precisely we study the application of the \( P - \delta \) controller given by:

\[
C(s) := k_p + k_\delta e^{-\tau s}, \tag{10}
\]

where \( k := [k_p, k_\delta]^T \) are the controller gains and \( \tau \) is a fixed time-delay. We aim to analyze the stability of the system through the closed-loop transfer function:

\[
T(s) = \frac{C(s)G_1(s)}{C(s)G_1(s) + 1}. \tag{11}
\]

**B. Stability Analysis**

As established above, we are interested in finding the stability regions in the \( (k_p, k_\delta) \)-parameters space considering a fixed delay-value \( \tau \). To this end, let us consider the open-loop transfer function \( G_1(s) \) (8), along with the \( P - \delta \) controller (10), leading to the closed-loop characteristic equation:

\[
\Delta(s) := \frac{LC}{V_s} s^2 + \frac{L}{V_s R s} + \frac{1}{V_s} + k_p + k_\delta e^{-\tau s} = 0. \tag{12}
\]

In order to simplify the analysis, in the remaining part of the report we will adopt the following notation:

\[
a := \frac{LC}{V_s}, \quad b := \frac{L}{RV_s}, \quad c := \frac{1}{V_s}. \tag{13}
\]

In this way, we can rewrite the characteristic equation as:

\[
\Delta(s) = as^2 + bs + c + k_p + k_\delta e^{-\tau s} = 0. \tag{14}
\]

It is well known that the stability of the closed-loop system is directly related to the location of the roots of (14) (see, [11], for further details). More precisely, the closed-loop system is stable if and only if all the roots of the characteristic equation are located in the LHP (Left-Half Plane) of the complex plane.

**Definition 1 (Frequency crossing set):** The frequency crossing set \( \Omega \subset \mathbb{R} \) is the set of all \( \omega \) such that, there exist at least a triplet \( (k_p, k_\delta, \tau) \) for which:

\[
\Delta(j\omega; k_p, k_\delta, \tau) = 0. \tag{15}
\]

**Remark 2:** It is clear that if we take the complex conjugate of (15), the following equality holds:

\[
\Delta(-j\omega; k_p, k_\delta, \tau) = \Delta(j\omega; k_p, k_\delta, \tau). \tag{16}
\]

Therefore, in the rest of the report we will consider only nonnegative frequencies, i.e., \( \Omega \subset \mathbb{R}_+ \cup \{0\} \).

**Definition 2 (Stability Crossing Curves):** The stability crossing curves \( \mathcal{T} \) is the set of all parameters \( (k_p, k_\delta, \tau) \in \mathbb{R}^2 \times \mathbb{R}_+ \) for which there exist at least one \( \omega \in \mathbb{R}_+ \cup \{0\} \) such that \( \Delta(j\omega; k_p, k_\delta, \tau) = 0 \). For a fixed delay value \( \tau^* \in \mathbb{R}_+ \), any point \( k \in \mathcal{T} \) is known as a crossing point.

1) **Delay-Independent Stability:** First, we analyze the stability of the closed-loop system considering any possible fixed delay value \( \tau \in \mathbb{R}_+ \), this particular result is shown as follows:

**Proposition 1:** Let \( a, b, c \in \mathbb{R}_+ \), then, the buck DC/DC converter is asymptotically stable independent of the delay value \( \tau \in \mathbb{R}_+ \), if the controller gains satisfy the following conditions:

\[
k_\delta^2 < \frac{b^2}{4a^2} (4ak_p + 4ac - b^2), \tag{16}
\]

\[
k_p > \frac{b^2}{4a} - c. \tag{17}
\]

2) **Stability Crossing Curves Characterization:** Now, in order to perform a stability analysis for a specific delay value in the controller design, we first need to construct the stability crossing boundaries. Then, it will be useful to characterize the behavior of the roots movement as a parameter variation crosses some of these boundaries. This section focuses on analyzing such a behavior.

**Proposition 2:** Let \( \tau \in \mathbb{R}_+ \) be a fixed value and \( \sigma, \omega \in \mathbb{R} \). Then, \( \Delta(s) \) has a root at \( s = \sigma + j\omega \), if and only if the controller gains \( k(\sigma, \omega) := [k_p(\sigma, \omega), k_\delta(\sigma, \omega)] \), are given as:

\[
\left\{ \begin{array}{c}
k_p(\sigma, \omega) = a(\omega^2 - \sigma) - (2a\sigma + b)\omega \cot(\tau\omega) - b\sigma - c, \\
k_\delta(\sigma, \omega) = (2a\sigma + b)\omega e^{\sigma \tau} \csc(\tau\omega), \end{array} \right. \quad \text{if } \omega \neq 0 \tag{18}
\]

\[
k_\delta = e^{\sigma \tau} [-k_p - (a\sigma^2 + b\sigma + c^)], \quad \text{if } \omega = 0. \tag{19}
\]

The stability crossing curves are characterized in the following result.

**Proposition 3:** Let \( \tau \in \mathbb{R}_+ \) be a fixed delay value and \( \Omega := \bigcup_{\ell} \Omega_\ell \cup \{0\} \) for \( \ell \in \mathbb{N} \), where the subsets \( \Omega_\ell \) are defined as:

\[
\Omega_\ell := \{ \omega \in \mathbb{R}_+ | \omega \in \left( \frac{\pi}{2}(\ell - 1), \frac{\pi}{2}\ell \right) \}. \tag{20}
\]

Then, \( \omega \in \Omega \setminus \{0\} \) is a crossing frequency if and only if \( k(\omega) := [k_p(\omega), k_\delta(\omega)] \), where:

\[
k_p(\omega) = a\omega^2 - c - b\omega \cot(\tau\omega), \tag{21}
\]

\[
k_\delta(\omega) = b\omega \csc(\tau\omega), \tag{22}
\]

defines a crossing point \( k(\omega) \in \mathcal{T} \). Moreover, the line

\[
k_\delta = -k_p - c, \tag{23}
\]

defines a stability crossing curve at \( \omega = 0 \).

Given all stability crossing points \( k(\omega) \) and the frequency crossing set \( \Omega \), we can define each stability crossing curve through its continuity as follows:

\[
\mathcal{T}_0 := \{ k \in \mathbb{R}^2 | k_\delta = -k_p - c \}, \tag{24}
\]

\[
\mathcal{T}_\ell := \{ k(\omega) \in \mathbb{R}^2 | \omega \in \Omega_\ell \text{ for } \ell \in \mathbb{N} \}. \tag{25}
\]

Finally, we describe the set \( \mathcal{T} \) as:

\[
\mathcal{T} = \bigcup_{\ell} \mathcal{T}_\ell, \quad \ell \in \mathbb{N} \cup \{0\}. \tag{26}
\]
3) Stability Region with a Fixed Delay: It is clear that if \( k_\delta(\omega) \neq 0 \) for \( \omega \in \Omega \) and \( \ell \in \mathbb{N} \), then the stability crossing curves \( \mathcal{T}_\ell \) does not cross the \( k_p \)-axis. Furthermore, the only curve that crosses the \( k_p \)-axis, precisely at \( k_p = 0 \) is \( \mathcal{T}_0 \), which is the line defined in (23). Now, notice that since the physical parameters \( L, C \) and \( R \) are positive, then \( a, b, c \in \mathbb{R}_+ \). Now, observing the sign of \( k_\delta(\omega) \), we can conclude that a stability crossing curve \( \mathcal{T}_\ell \) with \( \ell \) even or \( \ell \) odd is located above or below the \( k_p \)-axis, respectively. Finally, bearing in mind the above facts, it will be useful to introduce the following sets:

\[
\mathcal{B}_\ell := \left\{ k \in \mathbb{R}^2 \middle| \begin{array}{l}
\bar{k}_\delta < k_\delta; \quad \forall k \in \mathcal{T}_\ell \quad \text{for} \quad \ell \in 2\mathbb{N} \\
\bar{k}_\delta > k_\delta; \quad \forall k \in \mathcal{T}_\ell \quad \text{for} \quad \ell \in 2\mathbb{N} + 1 \\
\bar{k}_\delta < -\bar{k}_p - c \quad \text{for} \quad \ell = 0
\end{array} \right\}
\]

Hence, the sets \( \mathcal{B}_\ell \) are the collection of all points below and above the curves \( \mathcal{T}_\ell \) for \( \ell \) odd and \( \ell \) even, respectively, and the set \( \mathcal{B}_0 \) is the set of all points below the stability crossing curve \( \mathcal{T}_0 \). Finally, we have the following proposition:

**Proposition 4:** Given a fixed delay \( \tau \in \mathbb{R}_+ \), there always exists an open connected stability region \( \mathcal{H} \) defined by:

\[
\mathcal{H} := \bigcap_{\ell \in \mathbb{N}} \mathcal{B}_\ell \cap \mathcal{B}_0.
\]

Furthermore, \( \mathcal{H} \) is unbounded.

4) Crossing Directions Characterization: In order to compute a stability index which is the number of roots in the RHP for a given parametrical region it is of interest to determine the roots tendency as the vector \( k \) deviates from the curve \( \mathcal{T} \). The following results are the main tools applied in this work to achieve such a task.

**Proposition 5:** A pair of roots of the characteristic equation (14) moves from the LHP to the RHP as \( k \) crosses a stability crossing curve \( k(\omega) \) with \( \omega \neq 0 \) in the increasing direction of \( k \chi \) for \( \chi \in \{ p, \delta \} \) if:

\[
C_k \triangleq b(\tau \alpha \cot(\tau \omega) - 1) \cos(\eta_k \tau \omega) + \eta_k (b \tau + 2 \alpha) \omega \sin(\tau \omega) > 0,
\]

where the indicative function \( \eta_k \) is defined as:

\[
\eta_k := \left\{ \begin{array}{ll} 0, & \text{if} \quad \chi = p, \\ 1, & \text{if} \quad \chi = \delta. \end{array} \right.
\]

Furthermore, the crossing is from the RHP to the LHP if the inequality is reversed.

Observe that Proposition 10 does not give any information about the crossing when \( \omega = 0 \). The following result fills this gap.

**Proposition 6:** Given a fixed delay \( \tau \in \mathbb{R}_+ \), then, one root of (14) crosses from the LHP to the RHP of the complex plane through the origin as \( k \) crosses \( \mathcal{T}_0 \) from left to right if the intersection of \( k \) and \( \mathcal{T}_0 \) is located at the left of the point \( k_0 \in \mathcal{T}_0 \), defined by:

\[
k_0 := [k_{p,0}, k_{\delta,0}] = \frac{1}{b}[\tau b - c, b] = \frac{1}{b}. \]

Furthermore, the crossing of the root is from the RHP to the LHP if the intersection is located at the right of \( k_0 \).

C. \( \sigma \)-Stability and Fragility

In this section we propose two auxiliary results to deal with the problems of \( \sigma \)-stability and fragility of a given controller. These results will be useful in the design of a controller that satisfies a given performance (exponential decay rate) as well as some robustness against parametrical uncertainties. To this end, let us first state the \( \sigma \)-stability problem: Let \( \sigma \in \mathbb{R}_+ \), the \( \sigma \)-stability problem can be described as the task of determining a controller \( k \) such that the real part of the rightmost roots of the characteristic equation (14) is located at the left of \( \sigma \). Let \( \mathcal{T}_\sigma \) denote the set of all \( k \) such that (14) has at least one root on the vertical line of the complex plane defined as \( L_\sigma := \sigma + j\omega \) for all \( \omega \in \mathbb{R} \). This vertical line is defined as the \( \sigma \)-axis. In order to introduce similar results to those presented in Section III-B.2 we have the following:

**Corollary 1:** Let \( \omega \in \Omega \), and let \( \tau \in \mathbb{R}_+ \), \( \sigma \in \mathbb{R}_- \) be fixed values. Then, the set \( \mathcal{T}_\sigma \) can be computed as:

\[
\mathcal{T}_\sigma = \mathcal{T}_\sigma \cup \mathcal{T}_{\sigma,0},
\]

with

\[
\mathcal{T}_{\sigma,0} = \{ k \in \mathbb{R}^2 | k = k(\sigma, 0) \},
\]

\[
\mathcal{T}_\sigma = \{ k \in \mathbb{R}^2 | k = k(\sigma, \omega) \}.
\]

Consider now the fragility problem, which consists of computing the maximum controller parameters deviation \( d \) of a given stabilizing controller \( \bar{k} := [\bar{k}_p, \bar{k}_\delta]^T \), such that the closed-loop system remains stable, as long as the controller parameters \( k \) satisfy the inequality:

\[
\sqrt{(k_p - \bar{k}_p)^2 + (k_\delta - \bar{k}_\delta)^2} < d.
\]

In order to address this problem, let \( k(\omega) = [k_p(\omega), k_\delta(\omega)]^T \) as given in Proposition 3. Bearing in mind this notation, we have the following:

**Proposition 7:** Let \( \bar{k} \) be a stabilizing controller. Then, the maximum parameter deviation \( d \) of \( \bar{k} \), such that the closed-loop system remains stable, is given by:

\[
d := \min \left\{ \bar{d}, \frac{1}{\sqrt{2}} |\bar{k}_p + \bar{k}_\delta + c| \right\},
\]

with \( \bar{d} \) defined as:

\[
\bar{d} := \min_{\omega \in \Omega_f} \{ \xi(\omega) \},
\]

where \( \Omega_f \) denote the set of all roots of \( f(\omega) \):

\[
f(\omega) := k(\omega) - \bar{k} \frac{d}{d\omega} k(\omega).
\]

D. Illustrative Results

All results of this section were obtained by means of the “SimPowerSystems” toolbox in the “Simulink” environment of the software “Matlab”. The parameters used in the simulation are summarized in Table I. The tests presented in this section are designed to regulate the output voltage \( v_o(t) \) to a nominal value of \( V_o := 20 \). Recall that the control scheme has the task to regulate the variations of the output voltage \( \tilde{v}_o(t) \) to zero in order to satisfy the following: \( v_o(t) \to V_o \). The
control law proposed for the achievement of this objectives is given by:

\[ u(t) = U + \tilde{u}(t), \]  

(36)

with:

\[ \tilde{u}(t) := k_p e(t) + k_\delta e(t - \tau), \]  

(37)

where the error signal is defined as:

\[ e(t) := 0 - \bar{v}_o(t) = V_o - v_o(t), \]  

(38)

and the nominal value \( U \) can be obtained directly from (7).

Consider the controller’s delay value \( \tau = 5 \times 10^{-5} \text{s} \) for the

\begin{table}[h]
\centering
\begin{tabular}{c|c|c}
Symbol & Value & Unit
\hline
\( R \) & 3 & \( \Omega \) \\
\( L \) & \( 180 \times 10^{-5} \) & \( \text{H} \) \\
\( C \) & \( 40 \times 10^{-6} \) & \( \text{F} \) \\
\( V_o \) & 40 & \( \text{V} \) \\
\( f \) & \( 20 \times 10^3 \) & \( \text{Hz} \)
\end{tabular}
\end{table}

\( P-\delta \) controller (10) along with the parameters given in Table I. Figure 2, first depicts the delay-independent stability region obtained directly from Proposition 1. Second, it illustrates the stability region for the given fixed value \( \tau \) obtained by means of Proposition 4 and computed by applying Proposition 3. Making use of these results, the \( P-\delta \) controller is set as

\[ k = [50, 1]^T. \]  

Three different values of \( \tau \) are proposed in order to verify the independent stability condition. The results of the regulation of \( v_o(t) \) for this test are depicted in Fig. 3. In order to show how the stability region for the fixed value \( \tau = 5 \times 10^{-5} \text{s} \) behaves, a large view of this region is depicted in Fig. 4.

In the following, let us consider the stability index \( \eta \) (number of roots in the RHP) for different regions delimited by the stability crossing boundaries. To this end, Fig. 5 presents the results of applying Propositions 5 and 6, where the colors “red”, “green”, “blue” and “black” stands for \( C_p > 0, C_\delta > 0, C_pC_\delta > 0 \) and \( (C_p < 0) \& (C_\delta < 0) = \text{true} \), respectively.

Finally, we apply the auxiliary results shown in Section III-C. First we use a \( \sigma \)-Stability Analysis varying \( \sigma \) from 0 to \( -4 \times 10^3 \), the results are summarized in Fig. 6. From this figure, we denote the finding of a stability region that ensures that all roots of the characteristic equation of the closed-loop system has real part \( \sigma < -4 \times 10^3 \). In other words, that the exponential decay rate of the closed-loop system response is smaller than \( 4 \times 10^3 \). Second, we test the fragility for three different controllers, the results are summarized in Table II and illustrated in Fig. 7.
In mechanical systems, one of the classical problems of automatic control is the stabilization of the inverted pendulum on its unstable equilibrium point at the upright position. In this work we study this task in the case of the so-called Furuta pendulum (see, Fig. 8), also known as the rotatory inverted pendulum.

In order to achieve this task, we propose the use of some standard Linear Quadratic Regulation (LQR) controller. Some insights concerning this control law for three underactuated systems (inverted pendulum on a cart, inverted wedge and ball and beam system) can be found in [19]. Furthermore, we consider a time-delay in the state feedback loop which can be inherent to the system due to data processing, or even designed for performance requirements.

This section is organized as follows: Section IV-A concerns to the Furuta pendulum modelling and the LQR control design for the stabilization in the delay-free scenario. Section IV-B.1 shows a simple method easy to implement for computing the critical time-delay value in the state feedback loop at which the closed-loop system loses stability. In Section IV-B.2 the stability boundaries for two auxiliary gains and the time-delay are characterized. Section IV-C shows some numerical results obtained on the software “Matlab”. Finally, Section IV-D discusses some future work.

### IV. Mechanical Application: Controlled Furuta Pendulum with a Delayed State Feedback Loop

In mechanical systems, one of the classical problems of automatic control is the stabilization of the inverted pendulum on its unstable equilibrium point at the upright position. In this work we study this task in the case of the so-called Furuta pendulum (see, Fig. 8), also known as the rotatory inverted pendulum.

In order to achieve this task, we propose the use of some standard Linear Quadratic Regulation (LQR) controller. Some insights concerning this control law for three underactuated systems (inverted pendulum on a cart, inverted wedge and ball and beam system) can be found in [19]. Furthermore, we consider a time-delay in the state feedback loop which can be inherent to the system due to data processing, or even designed for performance requirements.

This section is organized as follows: Section IV-A concerns to the Furuta pendulum modelling and the LQR control design for the stabilization in the delay-free scenario. Section IV-B.1 shows a simple method easy to implement for computing the critical time-delay value in the state feedback loop at which the closed-loop system loses stability. In Section IV-B.2 the stability boundaries for two auxiliary gains and the time-delay are characterized. Section IV-C shows some numerical results obtained on the software “Matlab”. Finally, Section IV-D discusses some future work.

### A. Prerequisites on Control Scheme Design

This section includes the basics of the design of a LQR based control scheme in the delay-free scenario for the Furuta pendulum stabilization problem. It covers the Furuta pendulum modelling and the LQR controller gain tuning.

#### 1) Furuta Pendulum Modeling:

As discussed in the sequel, we introduce the Furuta pendulum nonlinear model and a linear representation valid uniquely around an operating point of interest.

Figure 8 depicts the representation of the Furuta pendulum, also known as the rotatory inverted pendulum. This mechanical system has two degrees of freedom and two rotatory joints. It consists in three essential components: a motor and two bars known as arm and pendulum. The motor’s shaft is fixed at one end of the arm inducing a rotatory movement of this bar. The pendulum is placed at the opposite end to the motor’s shaft with a rotatory joint which provides a free rotatory movement in a normal plane to the arm. As shown in the schematic representation illustrated in Fig. 8, $\theta_0$ and $\theta_1$ are the arm and pendulum angular positions, respectively. $\theta_0$ is measured with respect of the $X$-axis and...
\[ \theta_1 \] with respect to the upright position. \( \mathcal{T} \) concerns to the torque applied to the arm and it is provided by the motor. \( I_0 \) and \( J_1 \) stands for the motor-arm and pendulum inertia values and \( L_0 \) and \( l_1 \) represent the arm length and the pendulum's center of mass location, respectively. Finally, \( m_1 \) represents the mass of the pendulum, while \( g \) denotes the gravitational acceleration.

![Furuta Pendulum Diagram (20)](image)

As detailed in [20] (see also, [21] and [22]), the Lagrangian formulation of the Furuta pendulum consists in the following:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = F, \quad (39) \]

where:

\[ M(q) := \begin{bmatrix} I_0 + m_1(L_0^2 + l_1^2 \sin^2 \theta_1) & m_1 l_1 L_0 \cos \theta_1 \\ m_1 l_1 L_0 \cos \theta_1 & J_1 + m_1 l_1^2 \end{bmatrix}, \]
\[ C(q, \dot{q}) := \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}, \quad g(q) := \begin{bmatrix} 0 \\ -m_1 l_1 g \sin \theta_1 \end{bmatrix}, \]
\[ c_{11} := \frac{1}{2} m_1 l_1^2 \dot{q}_2 \sin(2q_2), \]
\[ c_{12} := -m_1 l_1 L_0 \dot{q}_2 \sin q_2 + \frac{1}{2} m_1 l_1^2 \dot{q}_1 \sin(2q_2), \]
\[ c_{21} := -\frac{1}{2} m_1 l_1^2 \dot{q}_1 \sin(2q_2), \]
\[ F := \begin{bmatrix} \mathcal{T} \\ 0 \end{bmatrix}, \quad q := \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}. \]

In this work we focus on the control problem of stabilization and regulation of the solution pair \((\theta_0(t), \theta_1(t))\) around an operating point. Inspired by [20], we consider the corresponding system's linearization:

\[ \dot{x} = Ax + Bu, \quad (40) \]

where the state of the system \( x \) and the control variable \( u \) are defined as:

\[ x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix}, \quad u := \mathcal{T}. \quad (41) \]

The constant matrices \( A \) and \( B \) are given by:

\[ A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \beta & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ \gamma \\ 0 \\ \epsilon \end{bmatrix}, \quad (42) \]

where:

\[ \alpha := \frac{-g m_1^2 l_1^2 L_0}{l_0(J_1 + m_1 l_1^2 + J_1 m_1 L_0^2)}, \]
\[ \beta := \frac{\frac{1}{2} m_1 l_1^2 L_0 \dot{q}_2 \sin q_2 + \frac{1}{2} m_1 l_1^2 \dot{q}_1 \sin(2q_2)}{l_0(J_1 + m_1 l_1^2 + J_1 m_1 L_0^2)}, \]
\[ \gamma := \frac{J_1 + m_1 l_1^2}{l_0(J_1 + m_1 l_1^2 + J_1 m_1 L_0^2)}, \]
\[ \epsilon := \frac{-m_1 l_1 L_0}{l_0(J_1 + m_1 l_1^2 + J_1 m_1 L_0^2)}. \]

Notice that this system is valid for any operation close to the unstable equilibrium point:

\[ \begin{bmatrix} \theta_0 \\ \theta_1 \\ \dot{\theta}_0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (43) \]

(that is, the pendulum is located at the upright position).

2) State Feedback Gain Design: As mentioned above, in order to stabilize the linear system (40) through a state feedback control law we use a similar approach to the one presented in [15]. This is known as the standard LQR control problem. The technique consists in computing the optimal solution for the linear quadratic cost functional:

\[ J = \inf_{u(t) \in L_2[0, \infty]} \int_0^\infty \left[ x(t)^T Q x(t) + r u(t)^2 \right] dt. \quad (44) \]

In this expression, the weights \( Q \geq 0 \) and \( r \geq 0 \) are chosen with the purpose of reducing the states \( x \) and the cost of the control \( u \). Qualitatively, if \( Q \) is a diagonal matrix, the position of the greater value of this matrix represents the most important state to be reduced. In the same manner, as \( r \) is chosen with a greater value, in such a way that the energy provided by the control law \( u \) must be lower.

The solution to the functional (44) is the state feedback control law:

\[ u(t) = -K^* x(t) = r^{-1} B^T P x(t), \quad (45) \]

where:

\[ K^* = [k_a, k_b, k_c, k_d] = -r^{-1} B^T P, \quad (46) \]

and \( P \) is the unique symmetric positive-defined solution to the Riccati equation:

\[ A^T P + PA - PB r^{-1} B^T P + Q = 0. \quad (47) \]
B. Computing Delay Margin and Retarded Gains Tuning

This section contains two stability analysis by considering a time-delay in the state feedback loop. First, we compute the critical delay value in the feedback loop for a proper choice of $K^*$. Second, we propose two auxiliary gains which will give us two degrees of freedom (2-DOF), allowing to improve the system’s response.

1) Delay Margin and Robustness Issues: It is well known that in a closed-loop system, if the control law is implemented by means of a digital platform, then, there always be present a time-delay due to the computational data processing. In this regard, the delay is a consequence of sensor with built-in data processing. In this section we aim to characterize this behavior by considering a time-delay in the control law.

Having designed the vector gain $K^*$ as shown in the previous section, we propose the following:

$$u(t) = -K^*x(t - \tau), \quad (48)$$

where $\tau > 0$ is a fixed delay value. Furthermore, one may notice that $\tau$ can be defined as $\tau := \tau_p + \tau_d$ where the time-delay values $\tau_p$ and $\tau_d$ refers to the data processing and, to a control design parameter, respectively.

Remark 3: It is well known that the stability of the closed-loop system is directly related to the location of the roots of the characteristic equation (see, [11], for further details). More precisely, the closed-loop system is stable if and only if all the roots of the characteristic equation are located in the LHP (Left-Half Plane) of the complex plane.

Since for $\tau = 0$ the closed-loop system is locally asymptotically stable around the origin, therefore, all of the roots of the characteristic equation given by:

$$\Delta_0(s) := \det \{sl - (A - BK^*)\} = 0, \quad (49)$$

have negative real parts. In other words, all of its roots remain in the LHP of the complex plane for a proper choice of $K^*$. Now, by taking into account the control law (48), the characteristic equation of the closed-loop system can be expressed as:

$$\Delta_\tau(s) = \det \{sl - [(A - BK^*)e^{-\tau s}]\} = 0, \quad (50)$$

or more compactly as:

$$\Delta_\tau(s) = P(s) + Q(s)e^{-\tau s}, \quad (51)$$

where:

$$P(s) = s^4 - \beta s^3,$$
$$Q(s) = (k_b \gamma + k_d e) s^3 + (k_d \gamma + k_c e) s^2 + k_b (\alpha e - \beta \gamma)s + k_d (\alpha e - \beta \gamma).$$

Remark 4: As mentioned by [11], this type of function ($\Delta_\tau(s)$) is known as a quasi-polynomial, one of its main differences with respect to a common polynomial, is that it has an infinite number of roots. Furthermore, the roots of $\Delta_\tau(s)$ move continuously with respect to variations of its parameters (coefficients, delay) and there is always a finite number of roots at the right side of any vertical line of the complex plane.

The appropriate computation of the critical delay value at which the closed-loop system loses stability is given below:

**Proposition 8:** The closed-loop system is asymptotically stable for any delay value $\tau \in [0, \tau_\text{c})$, where:

$$\tau_\text{c} = \min \{ \tau^* \in \mathbb{R} | \tau^*(\omega^*) > 0, \omega^* \in \Omega_p \}. \quad (52)$$

where:

$$\tau^*(\omega^*) = \frac{1}{\omega^*} \left[ \arg \left\{ \frac{Q(\omega^*)}{P(\omega^*)} \right\} + 2n\pi \right], \quad n \in \mathbb{Z}, \quad (53)$$

and where the set $\Omega_p$ as defined as the set of all real roots of the following equation:

$$|Q(i\omega^*)|^2 - |P(i\omega^*)|^2 = 0. \quad (54)$$

Proof: By taking into account Remark 4, and the fact that the closed-loop system is stable for $\tau = 0$ implies that for $\tau > 0$ sufficiently small all the roots of (51) will remain on the LHP of the complex plane. Moreover, there is a critical value $\tau$ such that (51) has at least one root on the imaginary axis and hence, such a value induces to the closed-loop system to lose stability.

As can be seen in [23], there exists a value $\tau$ such that the quasi-polynomial $\Delta_\tau(s)$ has a root on the imaginary axis in $s = i\omega^*$, if and only if, the following condition:

$$\left| \frac{Q(i\omega^*)}{P(i\omega^*)} \right| = 1, \quad (55)$$

holds for some value $\omega^* \in \mathbb{R}_+$. Moreover, the correspondent time-delay value can be computed by (53). Furthermore, notice that the condition (55) can be rewritten easily as (54), which is a polynomial, implying that it has a finite number of solutions. Finally, by defining $\Omega_p$ as the set of all real roots of (51), the critical delay value can be computed as in (52).

2) Extended Controller (2-DOF): In the previous section, we show a method for computing the margin delay in order to maintain stability in the closed-loop system. In this section, we propose a 2-DOF controller, where will be shown to be useful when the inherent delay in the system is larger than the critical delay computed above.

Let $K^*$ be a stabilizing gain for the delay-free scenario, computed using the results shown in section IV-A.2. We consider as our new state feedback gain:

$$K = K^* + [k_1, 0, k_2, 0], \quad (56)$$

where $k_1, k_2 \in \mathbb{R}$ are compensating gains in both positions ($\theta_0, \theta_1$) feedback loop. Considering this extended controller, the characteristic function of the closed-loop system is given by:

$$\Delta_\tau^*(s) = P(s) + (Q(s) + (\alpha k_1 + \epsilon k_2)s^2 + k_1 (\alpha e - \beta \gamma)) e^{-\tau s}. \quad (57)$$

**Remark 5:** It is worth noticing that the proposed state feedback gain (56) has a particular structure which provides two degrees of freedom in the positions regulation problem. We are interested in such structure, since the appropriate
regulation of position implies the regulation of velocity to the origin.

Now, we introduce some notation: let \( \rho(\omega) := \alpha e - \gamma(\alpha^2 + \beta) \) and, \( R(\omega) \) and \( I(\omega) \) be the real and imaginary part of \( Q(i\omega) \), respectively. Furthermore, it is worth noticing that \( P(i\omega) \in \mathbb{R} \), for all real \( \omega \). Having explained this approach, the following result characterize the triplet \((k_1, k_2, \tau)\) at which the system has at least one root on the imaginary axis.

**Proposition 9:** Let \( K^* \) be a stabilizing gain of the delay-free scenario and let \( \rho \), \( I \) and \( R \) be as defined previously. Then, the characteristic function \( \Delta_k^*(s) \) has at least two roots on the imaginary axis at \( s = \pm i\omega \), if and only if:

\[
\tau_d = \frac{1}{\omega} \sin^{-1} \left\{ -\frac{I(\omega)}{P(i\omega)} \right\}, \quad \forall \omega \in \Omega_d, \tag{57}
\]

where:

\[
\Omega_d := \left\{ \omega \in \mathbb{R} \mid \left| \frac{I(\omega)}{P(i\omega)} \right| \leq 1 \right\}, \tag{58}
\]

and the gains \( k_1 \) and \( k_2 \) belong to the family of lines:

\[
k_2 = \frac{1}{\cos^2(\omega)} \left\{ \rho(\omega)k_1 + R(\omega) - I(\omega)\cot(\omega) \right\}, \quad k_1 = -k_a, \tag{60}
\]

for any \( \omega \in \Omega_d \). Furthermore, it has a single root at the origin, if and only if:

\[
\rho(\omega) = 0 \quad \text{and} \quad \tau \in \mathbb{R} \quad \text{for any} \quad \omega \in \Omega_d.
\]

**Proof:** Consider the characteristic function \( \Delta_k^*(s) \), by setting \( s = r e^{i\omega} \) the following equations system is obtained:

\[
\Re \{\Delta_k^*(s)\} = 0, \quad \Im \{\Delta_k^*(s)\} = 0, \tag{61}
\]

by trying to solve this system for \( k_1 \) and \( k_2 \), the following is computed:

\[
\begin{bmatrix}
\rho(\omega) \cos(\tau\omega) & -\epsilon \omega^2 \cos(\tau\omega) \\
-\rho(\omega) \sin(\tau\omega) & \epsilon \omega^2 \sin(\tau\omega)
\end{bmatrix}
\begin{bmatrix}
k_1 \\
k_2
\end{bmatrix}
= r(\omega), \tag{62}
\]

where \( r(\omega) \) is a vector-valued function which can be easily deduced and for the sake of brevity is omitted. It is clear to see that the determinant of the matrix related to equation (62) is equal to zero and, therefore, does not have a unique solution.

However, we can rewrite (61) as:

\[
\ell(\omega) \cos(\tau\omega) + I(\omega) \sin(\tau\omega) + P(i\omega) = 0, \tag{63}
\]

\[
\ell(\omega) \sin(\tau\omega) + I(\omega) \cos(\tau\omega) = 0, \tag{64}
\]

where:

\[
\ell(\omega) = \rho(\omega)k_1 - \epsilon \omega^2 k_2 + R(\omega). \tag{65}
\]

By solving the system of equations formed by (63) and (64) for \( \ell(\omega) \), and consequently comparing the obtained expressions the following condition must be fulfilled:

\[
P(i\omega) \sin(\tau\omega) + I(\omega) = 0. \tag{66}
\]

On one hand, any pair \((\tau, \omega)\) satisfying condition (66) also induces to equations (63) and (64) to be equivalent. On the other hand, for every \( \omega \in \mathbb{R} \), there exist an infinite set of pairs \((k_1, k_2)\) along the line (64), which also solves (63).

The proof ends by solving \( \tau \) and \( k_2 \) from (66) and (64) and obtaining conditions (57) and (59), respectively. Furthermore, condition (60) can be verified simply by solving \( k_1 \) from \( \Delta_k^*(0) = 0 \) and the set \( \Omega_d \) is defined to consider only real solutions of (57).

**C. Illustrative Results**

For further details on the examples proposed in the sequel, we refer to the website 1. Such material is composed by a variety of animations of the Furuta Pendulum system and system response signals behavior. The support material is listed below:

A.1 Free Motion Behavior of the Furuta Pendulum with initial conditions:

\[
[\theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0)]^T = \left[ \frac{\pi}{4}, \frac{\pi}{4}, 0 \right]^T. \tag{67}
\]

A.2 Controlled Motion with a delay value \( \tau = 0 \).

A.3 Controlled Motion with a delay value \( \tau = 0.5\tau_c \).

A.4 Controlled Motion with a delay value \( \tau = 0.9\tau_c \).

A.5 Unstable Response of the Furuta Pendulum with a delay value \( \tau = \tau_c \).

A.6 Smooth Time Delay Variation of the System Transient Response from \( \tau = 0 \) to \( \tau = \tau_c \).

The parameters used in these simulations are taken from the experimental test bench studied in [20] and are summarized in Tab. III. The initial conditions settled for the following numerical results are chosen near the origin as:

\[
[\theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0)]^T = \left[ \frac{\pi}{10}, 0, \frac{\pi}{9}, 0 \right]^T.
\]

\[
\text{TABLE III}
\]

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>9.81</td>
<td>( \pi )</td>
</tr>
<tr>
<td>( l_1 )</td>
<td>( 129 \times 10^{-3} )</td>
<td>( m )</td>
</tr>
<tr>
<td>( L_0 )</td>
<td>( 155 \times 10^{-3} )</td>
<td>( m )</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>( 22.18 \times 10^{-3} )</td>
<td>( Kg )</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>( 184.50 \times 10^{-6} )</td>
<td>( Kg.m^2 )</td>
</tr>
<tr>
<td>( l_0 )</td>
<td>( 238.49 \times 10^{-6} )</td>
<td>( Kg.m^2 )</td>
</tr>
</tbody>
</table>

As stated in section IV-A.2, in order to compute the state feedback gain \( K^* \) we need to chose the weights \( Q \) and \( r \). We set \( r = 1 \) and:

\[
Q = 1 \times 10^{-4} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0.01 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0.01
\end{bmatrix}, \tag{68}
\]

with the purpose of giving more importance to the convergence of the states \( \theta_0 \) and \( \dot{\theta}_1 \) than to the angular velocities or the control effort. More precisely, we set the position correspondent values in \( Q \) one hundred times larger than the ones set for the angular velocities. Given this parameters, the state feedback gain is computed as:

\[
K^* = -\begin{bmatrix}
0.0100 & 0.0049 & 0.1755 & 0.0161
\end{bmatrix}^T. \tag{69}
\]

1https://furutablind.wixsite.com/furuta
Now, as mentioned in section IV-B.1, the proposed strategy is meant to find the critical delay in the state feedback loop such that the closed-loop system becomes unstable. Having calculated the gain value $K^*$ and according to Proposition 8 we use equation (54) to compute the set $\Omega_p$. This consists in one element $\omega^* = 30.88$. Second, we calculate the critical delay value $\tau_c = 0.0344s$ from expression (52).

We show the results obtained under these considerations. Fig. 9 exhibits the closed-loop system response under different time-delay values below the critical condition: $\tau = 0, 0.5\tau_c, 0.9\tau_c$. The results are illustrated from the nonlinear and linear model point of view. As can be seen in Fig. 9, as the time-delay value tends to approximate to the critical value $\tau_c$, the system response tends to have a more oscillatory behavior in both models. This can be explained from the linear model perspective since as $\tau \to \tau_c$, the rightmost root of the characteristic equation (51) approaches the imaginary axis. Illustrative animations of the Furuta Pendulum for this particular cases can be found in A.2, A.3 and A.4. Furthermore, A.6 corresponds to the continuous change in the transient response as $\tau \to \tau_c$.

Moreover, in Fig. 10, we present an unstable response of the closed-loop system by setting $\tau = \tau_c$. At this value, the characteristic equation of the closed-loop system has at least one pair of roots on the imaginary axis. One of the main features of this test is that the nonlinear model clearly loses stability against the linear model which behaves more similar to a marginally stable system. The behavior of the Furuta Pendulum in this conditions is illustrated in A.5.

Finally, we use the result shown in Proposition 9 to compute the stability boundaries for the auxiliary gains $k_1$ and $k_2$. The results are depicted in Fig. 11, notice that since $K^*$ is a stabilizing gain for the delay-free scenario then, the origin of this figure illustrated by A is a stable point. Moreover, by remark 4 any variation of the parameters $(k_1, k_2, \tau)$ around the origin without crossing any stability boundary is stable. Therefore, the region around the origin delimited by the stability boundaries is a stable region. The point B depicts the parameters setting $(0, 0, \tau_c)$ which corresponds to the boundary of the previous analysis. As expected, this point lays on the stability boundaries. Furthermore, the line from $A$ to $B$ corresponds to the test illustrated in Figs. 9 and 10.

In order to verify this result, in Fig. 12 we explore the region in which $k_1 = 0.005$ illustrating the critical delay $\tau_c$. Furthermore, considering a larger delay $\tau = 0.04$, we test this regions by setting the parameters $p_1$, $p_2$ and $p_3$ in the closed-loop system with initial conditions:

$$\begin{bmatrix} \theta_0(0), \dot{\theta}_0(0), \theta_1(0), \dot{\theta}_1(0) \end{bmatrix}^T = \begin{bmatrix} \pi/20, \pi/20, 0 \end{bmatrix}^T.$$ 

The results are shown in Fig. 13, where is clear to see that the system has a stable response when $p_1$ and $p_2$ are chosen and an unstable response for $p_3$.

D. Future Work

We are interested in study two different design features for the Furuta pendulum stabilization problem, these are discussed in the sequel:

First, we aim to solve the problem of designing a control law such that the system can be driven from its stable equilibrium point in rest, to at least near the unstable point at the upright position. Moreover, considering a time-delayed nature in such a control law.

Second, in order to use the least requirements, we are interested in designing a $P - \delta$-controller type as in [24],
which does not require the full state. Moreover, we aim to attack this problem from a different point of view as made in past works in [25], [26] considering also the controllers fragility.

V. CONCLUDING REMARKS

A methodology for the design of a $P-\delta$ controller applied to the stabilization of a buck DC/DC converter is presented. In addition, the behavior of the roots of the characteristic equation, as the controller gains are varied is analyzed. The results go beyond the stabilization problem, particularly, the closed-loop performance analysis via the solution to the $\sigma$-stability problem and the study of controller fragility are addressed. This work was accepted and presented in the 25th Mediterranean Conference on Control and Automation under the title: “Closed-Loop Stability Analysis of Voltage Mode Buck Using a Proportional-Delayed Controller”.

The design of a LQR based control scheme considering a time-delay value in the state feedback loop for the stabilization of the Furuta pendulum is addressed. Furthermore, support didactic material in form of animations of this control scheme is also addressed. This analysis is submitted for revision for the XIX Congreso Mexicano de Robótica under the title: “Delay Margin in Controlling a Furuta Pendulum”.

In both applications, the design methodology can be applied and developed straightforwardly, showing that the presented results are easy to implement.

VI. ACTIVITIES SCHEDULE

1.-Bibliographic analysis and research.
2.-Academic courses.
3.-Development of a robustness criterion under external disturbances for delayed LTI systems.
4.-Study of stability analysis for delayed nonlinear systems based on Lyapunov functions.
5.-Study of delayed controllers applied to nonlinear systems.
6.-Development of a stability analysis and performance for non-linear systems.
7.-Thesis document writing
8.-Research internship on the laboratory of signals and systems in the university of Paris-sur.
9.-Experimental Tests.
10.-Previous exam.
11.-Final exam.

<table>
<thead>
<tr>
<th>Activity/Semester</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>3</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>4</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>7</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE IV

ACTIVITIES SCHEDULE

REFERENCES


