An Enhanced Process for the Fast Periodic Steady State Solution of Nonlinear Systems by Poincaré Map and Extrapolation to the Limit Cycle

J. Segundo-Ramírez¹, A. Medina²
Facultad de Ingeniería Eléctrica, División de Estudios de Postgrado U.M.S.N.H., Morelia, Michoacán, México

Abstract

This paper introduces an algorithm which dramatically reduces the computer effort required for the identification process of the transition matrix used for the fast steady state solution in the time domain on nonlinear power systems by extrapolation to the limit cycle. It is demonstrated that the proposed Enhanced Numerical Differentiation (END) Newton method increases the computer efficiency in at least hundred percent, when compared against the original Numerical Differentiation (ND) method and above 350 times, when compared against a conventional Brute Force (BF) method. The reported results are validated against the response obtained with a digital implementation with Simulink.

1. Introduction

The periodic steady state solution of nonlinear power networks can be obtained in the time domain by any conventional numerical integration method, such as the Fourth-order Runge-Kutta method (RK4) or the Trapezoidal Rule, respectively, once the initial transient dies-out [1]. The last widely used in the electromagnetic transient programs of the EMTP type. However, the convergence to the periodic steady state solution through these conventional Brute Force (BF) methods [1] is usually very slow, in special in poorly damped systems [2].

The potential and efficiency of the Newton method based on the ND for the fast periodic steady state solution in the time domain of nonlinear power networks by extrapolation to the limit cycle has been reported [3-8]. However, an important problem persists during the practical application of this Newton methods; it consists on the fact that the computational effort required for the identification process, column-by-column, of the square state transition matrix Φ may be very high, as for each identification of Φ the numerical integration over one period of time is needed n+1 times, where n is the number of state variables.

This paper proposes an Enhanced Numerical Differentiation method for the efficient periodic steady state solution of nonlinear power systems based on Poincaré Map and extrapolation to the limit cycle. The efficiency of the original ND method [3-8] is improved by at least hundred percent. As a consequence, the overall CPU time or number of full cycles needed by the Newton method based on the ND approach for the periodic steady state solution of nonlinear systems is substantially reduced, thus an Enhanced Numerical Differentiation (END) method is obtained. A Case study is presented to illustrate the practical application of the proposed END Newton method to the computation of the periodic steady state solution of nonlinear power systems. The efficiency of the END method is compared against the BF method and the ND method.

2. Extrapolation to the Limit Cycle

Let us assume the following nonlinear time-varying system periodically excited with a function of period T.

1 j2ramirez@gmail.com
2 amedina@umich.mx
\[
\frac{dx(t)}{dt} = f(t, x), \quad x(t_0) = x_0
\]  

(1)

where \( x \) and \( f \) are \( n \)-dimensional vectors, and \( f(t, x) \) is continuous and has a continuous first derivative with respect to \( x \). We assume that \( f(t, x) \) is periodic in \( t \) of period \( T \), so \( f(t_0) = f(t_0 + T) \). Also, it is assumed (1) has a periodic steady state solution \( \phi(t+T) = \phi(t) \), where \( \phi \) is the orbit of the system (1) in the limit cycle. Since the solution of (1) is assumed periodic, then it can be found by using the Poincaré map to extrapolate the state variables to the limit cycle through the application of a Newton method [3-9]. The process to compute the state variables \( x^\infty \) at the limit cycle is summarized next.

### 2.1 Newton Method

The dynamics in the neighborhood of the limit cycle can be approximated as

\[
\frac{d\Delta x(t)}{dt} = J(t)\Delta x(t)
\]

(2)

where \( J(t) \) is the Jacobian of \( f(t, x) \), and is defined as,

\[
J(t) = J_x(f(t, x))
\]

(3)

The solution of (2) expressed in discrete times, multiples of \( T \) is

\[
\Delta x_{k+1} = \Phi_k \Delta x_k
\]

(4)

where

\[
\Delta x_k = \Delta x(kT)
\]

(5)

Being \( k \) a integer number, \( \Phi_k \) is the discrete transition matrix of (2), which defines the sensitivity of the final state with respect to the starting state if we integrate over one period \( T \). In this extrapolation method, \( T \) is the period of the input signal, \( x_K \) is the state vector at the cycle \( K \); it can be mathematically expressed as,

\[
x_K = x_n + \Delta x_n + \Delta x_{n+1} + \ldots + \Delta x_{K-1}
\]

(6)

Equation (6) can be graphically interpreted as shown by Fig. 1. Notice that the periodic solution behaves like a fixed point solution in the Poincaré section.

From (4) we can obtain the following expression,

\[
\Delta x_m = (\Phi_{m-1}\Phi_{m-2}\ldots\Phi_n)\Delta x_n, \quad m > n
\]

(7)

![Fig. 1: Poincaré Section.](image)

Limit Cycle

Poincaré Section

\( x_n \)

\( x_{n+1} \)

\( x_{n+2} \)

\( x_{n+3} \)

\( x_{n+\infty} \)
For our purpose it is possible to assume that in the neighborhood of the limit cycle the following approximation of $\Phi_k$ applies,

$$\Phi_k = \Phi_{k+1} \approx \Phi_{k+2} \approx \cdots \approx \Phi$$  \hspace{1cm} (8)

Using (8), the expression (7) takes the simple form,

$$\Delta x_m = \Phi^{m-n} \Delta x_n$$  \hspace{1cm} (9)

Substituting (9) in (6) yields,

$$x_K = x_n + (I + \Phi + \cdots + \Phi^{K-1-n}) \Delta x_n$$  \hspace{1cm} (10)

Equation (10) can be expressed as follows

$$x_K = x_n + (I - \Phi)^{-1} (I - \Phi^{K-n}) \Delta x_n$$  \hspace{1cm} (11)

If we want to calculate the steady-state solution $x^\infty$, we must skip an infinite number of cycles. To get an appropriate formula we have to take,

$$x_c = \lim_{K \to \infty} x_K = x_i + (I - \Phi)^{-1} (x_{i+1} - x_i)$$  \hspace{1cm} (12)

where,

$$\Phi = \frac{\partial x(t+T)}{\partial x(t)}$$  \hspace{1cm} (13)

$x^\infty$ state variables at the limit cycle;

$x^i$ state variables at the beginning of base cycle;

$x^{i+1}$ state variables at the end of the base cycle;

$\Phi$ state transition matrix;

$I$ identity matrix.

The expression (12) corresponds to the algorithm based on the Poincaré map and the Newton-Raphson method. This expression can be seen as a special case of (11) for $K$ equal to infinite. In addition, (11) can be seen as a damped Newton method. The Newton method is suitable for computing unstable and stable limit cycles as well [9].

Different ways to compute the state transition matrix have been proposed [3-9]. The ND method [3-8] has been widely used, in merit to its simplicity, since it does not require the explicit state space formulation of the system nor the Jacobian $f$, as indicated in [9], but only the state vectors at the end and at the beginning of the cycle. However, this method may be associated with a prohibitive computational cost, especially in large electric systems. In such cases, the ND method may require more cycles to reach the limit cycle than the FB method if the system is sufficiently damped, even with a single identification of $\Phi$ [4].

In the next Section, the ND method is reviewed, then, the proposed Enhanced Numerical Differentiation (END) method is detailed.

2.2 Enhanced Extrapolation to the Limit Cycle

A. Numerical Differentiation Method: Review

The state transition matrix expressed by (13) can be approximated through finite-difference derivatives as

$$\Phi \approx \frac{\Delta x(t+T)}{\Delta x(t)}$$  \hspace{1cm} (14)

This matrix can be interpreted as a sensitivity matrix, since it gives the evolution of the perturbation one cycle later. Note that the $j$th column of $\Phi$ is $\Phi_j$; this column can be computed by perturbing the $j$th state, e.g., let $x(t) \rightarrow x(t) + \varepsilon x_j(t)$ and compute $x(t+T) \pm \Delta x_j(t+T)$, thus, $\Phi_j$ can be computed with (4). All $n$ states of the systems must be separately perturbed in order to compute the $n$ columns of the sensitivity matrix. Note that $n+1$ cycles must be computed before we can apply (12). $\Delta x_j(t)$ is the product of the $j$th column of the identity matrix times $\varepsilon$, where $\varepsilon$ is a small number ($10^{-6}$). $x(t+T)$ is computed by integrating from the state vector $x(t)$ over one period $T$; $x(t+T) \pm \Delta x_j(t+T)$ is computed by integrating from the state vector...
\( x(t) = \Delta x_j(t) \) over one period \( T \). This procedure for approximating \( \Phi \) in [3] is called ND method in [4].

If the perturbation is added to the state vector \( x(t) \), then the finite-difference derivative corresponds to the forward-difference formula, otherwise it is the backward-difference formula.

### B. Enhanced Numerical Differentiation Method

The proposed END method is oriented to the accelerated time domain periodic steady state solution of nonlinear power systems; it takes advantage of the half-waveform symmetry of excitation signals, such as voltage sources.

This methodology basically consists on the evaluation of (13), by the approximation of \( x(t+T) \) through the extrapolation of \( x(t+T/2) \). With this approximation, the integration of (1) for the computation of \( x(t+T) \) is not required to be done over a full period \( T \), as it is usually the case [3-8], but only over a half period, thus increasing the computational efficiency of the ND method at least in 100%.

Typically, in power electric systems, there are several operating scenarios. The simplest case is when the electric power network is balanced and free of harmonic distortion. Under this condition, the dc signals only have a dc component in its harmonic spectrum, and the ac components only have the fundamental frequency component. The periodic steady state solution of this type of systems can be found by using the phasor concept and a power flow algorithm.

A second case is when the electric power network is balanced and harmonic distorted. In this operating condition, the dc and ac signals have different harmonic components. The harmonic distortion is produced by the interaction between the power network and nonlinear loads and components, such as saturated transformer inductances, electric arc furnaces, and power electronic devices, among others. Due to the nature of the nonlinear inductances in power transformers, the harmonic components in the ac signals are odd, meanwhile in the dc signal the harmonic components are even. For the case of power electronic converters, the harmonic components in their ac side are usually odd and the harmonic components in the dc side are even. The power electronics converter can also produce even harmonic components in their ac side by changing the frequency of the carrier waveform employed in the modulation technique; however, in practice the index modulation is chosen to generate only odd harmonic component in the ac side of the converters and consequently even harmonics components in the dc side [13].

A more realistic case is when the voltage sources, the load, and the power system are unbalanced and harmonic distorted. However, the characteristic harmonic components in the dc and the ac signal are the same of the second case. Therefore, it can be generalized that the ac signal in the limit cycle satisfies the equation:

\[
x_j(t+T) = -x_j(t+T/2)
\]

The dc signals cannot satisfy (15), since in practical power systems they have only even harmonics. Therefore, it is equivalent to say that if a signal satisfies (15), then this signal does not have a dc component. On the other hand, the dc signals in the limit cycle satisfy the following equation:

\[
x_j(t+T) = x_j(t+T/2)
\]

Therefore, if a signal satisfies (16) in the limit cycle, then it is possible conclude that this is a dc signal. Furthermore, this dc signal satisfies (17).

\[
\frac{dx_j(t)}{dt} \frac{dx_j(t+T/2)}{dt} > 0
\]

Summarizing, the proposed END method is based on the following steps: 1) To integrate the system (1) from \( t \) to \( t+T/2 \); 2) To evaluate (17); for instance using finite differences approximations of its derivatives. This is a simple and fast operation, since it is performed only twice during the solution process. To evaluate (17) needs of a very simple operation and it is not a time consuming task. If (17) is not satisfied then proceed to integrate the system (1) from \( t+T/2 \) to \( t+T \) and use the conventional ND method. On the other hand, if (17) is satisfied,
identify the dc and ac variables. To do this, it is possible to say that in the neighborhood of the limit cycle, the ac variables satisfy the following equation,

\[ x_i(t+T) \approx -x_i(t+T/2) \] (18)

and, the dc variables satisfy the following equation,

\[ x_i(t+T) \approx x_i(t+T/2) \] (19)

The approximation of the state variables for the END Method is summarized in Fig. 2.

3. Initialization of the Newton method

The Newton method states that if the initial guess is close to a correct solution, then the Newton algorithm will converge with a quadratic rate. However, these methods only perform well if \( f \) is a smooth function, preferably differentiable with a Lipschitz continuous derivative [14]. Thus, it is important to provide a good initial guess for the starting of the Newton method.

Here, for the network initialization, a fundamental frequency phasor solution is provided; fundamental frequency models have to be used if they are available. In case of saturable elements such as nonlinear inductances, electric machines, transformers, amongst others, a linear characteristic can be used.

Once the linear solution has been computed, a number of full time domain cycles are first run using the detailed model of each electric component, thus allowing the initial transient and harmonics to settle down. The number of full cycles initially depends on the system characteristics, particularly damping level. In the authors’ experience, 10 cycles are sufficient for a well damped system with switching.

4. Simulation results

4.1 Three-Phase Electric System Including the Unified Power Flow Controller (UPFC)

In order to prove the proposed method, the test circuit including a Unified Power Flow Controller (UPFC) shown in Fig. 3 is used. This system contains two three-phase transmission lines between two voltage sources with a fundamental frequency of 60 Hz, and a shift angle of zero. The UPFC is represented with the model proposed in [10]. It includes the series control proposed in [11], and the shunt control proposed in [12]. Moreover, the UPFC model has the switching function explicitly represented. The advantage of this model is its applicability to various forms of pulse-width modulation or to other switching strategies. In this contribution, the sinusoidal pulse width modulation (SPWM) technique is used [13]. The series and shunt transformers are represented.
through \( RL \) branches.

The reference voltage vector for the series converter is calculated as [11],

\[
\begin{bmatrix}
  v_{ad}^r \\ v_{aq}^r
\end{bmatrix} = \begin{bmatrix}
  K_{ir} & -K_{p,sq} - K_{q,sq} / s \\
  K_{p,sp} + K_{q,sp} / s & K_{sp}
\end{bmatrix} \begin{bmatrix}
  i_{ad}^r \\ i_{aq}^r
\end{bmatrix} = \begin{bmatrix}
  v_{ad}^r - i_{ad} \\ v_{aq}^r - i_{aq}
\end{bmatrix}
\]

where \( i_{ad}^r \) and \( i_{aq}^r \) are the active and reactive reference currents, \( v_{ad}^r \) and \( v_{aq}^r \) are reference voltages of the series converter, respectively. The active and reactive reference currents are obtained from the active and reactive power flows and by measuring the voltage at the receiving end.

A two-stage control loop scheme is employed for the shunt converter of the UPFC. This scheme has two objectives: to control the voltage across the dc capacitor, and to regulate the ac voltage of the power system bus where the shunt converter is connected. The control scheme is given by

\[
\begin{bmatrix}
  m_{sh} \\ \alpha_{sh}
\end{bmatrix} = \begin{bmatrix}
  K_{pm} + K_{im} / s & 0 \\
  0 & K_{pm} + K_{is} / s
\end{bmatrix} \begin{bmatrix}
  v_{Pp}^r \\ v_{Qp}^r
\end{bmatrix} = \begin{bmatrix}
  v_{Pp}^r - v_{Pp} \\ v_{Qp}^r - v_{Qp}
\end{bmatrix}
\]

where \( v_{Pp}^r \) is reference magnitude of the shunt bus, \( v_{Pp} \) is the instantaneous magnitude of the shunt bus, \( m_{sh} \) is the index modulation ratio of the shunt converter and \( \alpha_{sh} \) is the phase angle shunt voltage.

The series controller regulates the real \((P_{ref}=0.45 \text{ pu})\) and reactive \((Q_{ref}=0 \text{ pu})\) power flows by adjusting the injected series voltage. The shunt converter regulates the dc-side capacitor voltage \((v_{dc}=2 \text{ pu})\) and the sending end voltage \((v_{Pp}=0.96 \text{ pu})\). The modulation index used is \(m_f=9\). The limit cycle is reached once a maximum absolute error criterion in the state variables is within \(10^{-10} \text{ pu}\). The test system dynamics are represented by a set of 26 ODEs. An integration step of 0.1 \(\mu\text{s}\) has been used in Simulink, and 33 \(\mu\text{s}\) and 16.3 \(\mu\text{s}\), respectively, for the proposed model. The fourth-order Runge-Kutta numeric integration method was used.

Fig. 3: Single-phase test system
The voltage $v_i$ is harmonic distorted with third and fifth components, this voltage is given by

$$
\begin{bmatrix}
  v_{ia} \\
  v_{ib} \\
  v_{ic}
\end{bmatrix} =
\begin{bmatrix}
  \cos(\omega t) + 0.03\cos(3\omega t) + 0.013\cos(5\omega t) \\
  \cos(\omega t - 2\pi/3) + 0.03\cos(3\omega t - 2\pi/3) + 0.013\cos(5\omega t - 2\pi/3) \\
  \cos(\omega t + 2\pi/3) + 0.03\cos(3\omega t + 2\pi/3) + 0.013\cos(5\omega t + 2\pi/3)
\end{bmatrix} \text{ pu}
$$

(21)

In addition, the transmission line connected with the series converter of the UPFC is unbalanced, with $Z_{ia} = 0.016 + j1.15 \text{ pu}$, $Z_{ib} = 0.017 + j1.15 \text{ pu}$, and $Z_{ic} = 0.018 + j1.15 \text{ pu}$.

The comparison of CPU times between the proposed method and SIMULINK cannot be done since the proposed method and this simulator are developed in different platforms. However, the comparisons can be carried out using the number of full cycles as a common unit of time instead of seconds. In this case, as the integration step has to be different, the computational efficiency must take into account the ratio of the integration step.

1) Convergence to the limit cycle

In order to make the damping of the system very low, e.g. with a Floquet multiplier of 0.99, the gains set has been selected as $K_{\alpha}=0.566$, $K_{S\alpha}=1$, $K_{S\beta}=0.0029$, $K_{S\gamma}=1$, $K_{S}=0.0029$, $K_{pm}=0.001$, $K_{p}=0.5$, $K_{m}=0.0005$, $K_{m}=0.1$.

Table I shows the results obtained in terms of number of full cycles of time (FC) required to obtain the periodic steady state solution using two different integration step, e.g. $\Delta t=33 \mu s$ and $\Delta t=16.3 \mu s$. In both cases, the performance of the BF, ND and the END is almost the same. The small difference is because the same ODE set is mapped at a different discrete-time system for each integration step. In both cases, the periodic steady state is achieved after 10495 periods using the BF approach based on the RK4 method, after 94 cycles when the ND method is applied, and after 38 cycles when the END method is applied. For $\Delta t=33 \mu s$, the END method is 2.82, and 356 times faster than the ND method, and the BF method, respectively. For $\Delta t=16.3 \mu s$, the END method is 2.84, and 361 times faster than the ND method, and the BF method, respectively. In addition, for both cases, the END and the ND methods converge in two and three iterations, respectively. The results in Table I show that it is more efficient to use the END, since only 38 full cycles are required to locate the limit cycle. In this particular case, the ND method needs one more iteration as compared with the ND method. However, in most cases the END and the ND methods require the same iterations to meet the specified criterion error.

Figure 4 shows the computed steady state solution for two different integration steps using the END method, e.g. $\Delta t=33 \mu s$ and $\Delta t=16.3 \mu s$, respectively, and an integration step of $\Delta t=0.1 \mu s$ for Simulink. Figures 4(a) and 4(b) show the dc voltage and the sending end voltage of phase A, respectively. Please notice that these variables achieve their reference values, e.g. 2 pu for the dc voltage capacitor and 0.96 pu for the sending end voltage, respectively. The results show to be in good agreement. It should be remarked that the END method represents a remarkable computational advantage over the BF method, and even over the ND method.

2) Harmonic

A close agreement between the Simulink and the END method solutions is obtained, as illustrated in Fig. 5(a) and 5(b), which show the harmonic distortion produced in the dc voltage and the sending end voltage of phase A, respectively. In this Figure, it can be noticed that dc voltage capacitor has only even harmonic components, not only multiple of 6. On the other hand, the terminal voltage and the series current contain only odd harmonic components [13]. Please notice that the spectrum obtained using the END method is in very close agreement for the two different integration steps, and a smaller integration step is needed when the solution is computed with Simulink.
Fig. 4: Steady-state solution comparison between the END method and Simulink

Table 1
Mismatches During Convergence of the BF, ND and END Methods

<table>
<thead>
<tr>
<th>FC</th>
<th>( \Delta t = 33 \ \mu s )</th>
<th>( \Delta t = 16.3 \ \mu s )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>ND</td>
<td>END</td>
</tr>
<tr>
<td>1</td>
<td>2.00\times10^{-1}</td>
<td>2.00\times10^{-1}</td>
</tr>
<tr>
<td>2</td>
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<td>10495</td>
<td>9.91\times10^{-11}</td>
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</table>
5. Conclusion

A Newton method based on an Enhanced Numerical Differentiation approach has been proposed for the fast periodic steady state solution in the time domain of nonlinear electric networks using a Poincaré map and extrapolation to the limit cycle process.

The proposed END method has been successfully applied for the computation of the periodic steady state solution of nonlinear-switched electric systems under severe operating conditions, e.g. very poor damping, phase unbalance, and harmonically distorted voltage sources.

The application END method approximately halved the number of cycles and, therefore, the computation effort needed by the ND method to reach the limit cycle and thus the periodic steady state; even though the ND and END methods have a quadratic rate of convergence.

It has been observed, for the analyzed case studies, that the END method is on average 356 and 2.8 times faster than the BF and ND methods, respectively.

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7. References

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Juan Segundo-Ramirez was born in Lázaro Cárdenas, Michoacán, México in 1979. He received his first degree in electromechanical engineering from the Instituto Tecnológico de Lázaro Cárdenas in 2001 and obtained his MSc Degree from the CINVESTAV Guadalajara in 2004. He is at present a Ph.D. candidate at the División de Estudios de Posgrado, Facultad de Ingeniería Eléctrica of the UMSNH. His area of research is dynamic and steady state analysis of power systems.

Aurelio Medina obtained his Ph.D. from the University of Canterbury, Christchurch, New Zealand in 1992. He has worked as a Post-Doctoral Fellow at the Universities of Canterbury, New Zealand (1 year) and Toronto, Canada (2 years). At present he is a staff member of the Facultad de Ingeniería Eléctrica, UMSNH, Morelia, Mexico where he is the Head of the Division for Postgraduate Studies. He is Senior Member of IEEE. His research interests are in the dynamic and steady state analysis of power systems.