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Design and Optimization of Delay-Based Controllers for Dynamical Systems

to obtain the Ph.D. degree in Electrical
Engineering

Julián Alejandro Hernández Gallardo

Universidad Autónoma de San Luis Potosí
Facultad de Ingeniería
Centro de Investigación y Estudios de Posgrado

San Luis Potosí, México, September 2026



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18 de junio de 2026

**MIE. JULIÁN ALEJANDRO HERNÁNDEZ GALLARDO
P R E S E N T E.**

En atención a su solicitud de Temario, presentada por los **Dres. César Fernando Francisco Méndez Barrios y Emilio Jorge González Galván**, Asesor y Coasesor de la Tesis que desarrollará Usted con el objeto de obtener el Grado de **Doctor en Ingeniería Eléctrica**, me es grato comunicarle que en la sesión del H. Consejo Técnico Consultivo celebrada el día 18 de junio del presente año, fue aprobado el Temario propuesto:

TEMARIO:

"Design And Optimization of Delay-Based Controllers for Dynamical Systems"

1. Introduction.
 2. Mathematical Preliminaries.
 3. Stability and Optimization of PI Controllers for Time-Delay Systems.
 4. Analysis and Design of Delay-Based Controllers for Delay-Free Systems.
 5. Delay-Based Controllers for Time-Delay Systems.
 6. Low-Level Control of an Underwater Drone.
- Concluding remarks
A. Laplace transforms
References.

"MODOS ET CUNCTARUM RERUM MENSURAS AUDEBO"

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"Diseño y Optimización de Controladores Basados en Retardo para Sistemas Dinámicos"

1. Introducción.
 2. Preliminares Matemáticos.
 3. Estabilidad y Optimización de Controladores PI para Sistemas con Retardo.
 4. Análisis y Diseño de Controladores Basados en Retardo para Sistemas sin Retardo.
 5. Controladores Basados en Retardo para Sistemas con Retardo Temporal.
 6. Control a Bajo Nivel de un Dron Submarino.
- Conclusiones.
- A. Transformadas de Laplace.
- Referencias.

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FACULTAD DE
INGENIERÍA



DOCTORAL THESIS

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Design and Optimization of Delay-Based Controllers for Dynamical Systems

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Study Programme Ingeniería Eléctrica

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Short abstract	This thesis addresses two fundamental problems in the field of dynamical systems stability. The first problem concerns the stabilization of delay-free dynamical systems using delay-based controllers. Although traditional control approaches typically avoid introducing delays, this work explores how intentional delay elements, when appropriately designed, can enhance system performance and robustness. Through analytical methods and simulations, the conditions under which such delay-based controllers achieve stability in delay-free systems are investigated. The second problem focuses on the stabilization of time-delay systems using low-order controllers. High-order controllers, although effective, can be impractical due to implementation complexity and sensitivity to model uncertainties. This research proposes control strategies that maintain system stability with reduced controller order, making them more suitable for real-world applications, as are PID controllers. Finally, everything shown in the simulation is validated on experimental platforms such as a DC motor and DC-DC converter.
Keywords	Time-delay systems; Delay-based control; Stability analysis; D-decomposition; Spectral abscissa; σ -stability; Controller optimization;

Declaration of Authorship

I hereby affirm that this doctoral thesis was prepared independently by me, using only the references, literature, and other professional sources cited herein.

Place & Date

Julián-Alejandro HERNÁNDEZ-GALLARDO



Dedication

To my father (†), my mother and Daniela, for all the love and support.

Para mi padre (†), mi madre y Daniela, por todo el amor y apoyo.

Julián



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First and foremost, I want to thank my parents. *Gracias, mamá, por tu amor incondicional, tu fuerza y tu fé constante en mí.* Your words and prayers have carried me through this journey more than you know.

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This PhD is not just a personal achievement — it's a reflection of all of you. *Esto también es suyo.*



Abstract

This thesis addresses the analysis, design, and optimization of delay-based controllers for dynamical systems with and without inherent time delays. The proposed methodologies are developed from a frequency-domain perspective and supported by quasi-polynomial analysis, stability crossing boundaries, and optimization-based procedures. The main objective is to demonstrate that intentional delays can be systematically incorporated into controller structures to improve closed-loop stability, transient performance, robustness, and exponential convergence properties.

First, proportional–integral (PI) controllers for systems with inherent time delays are studied. Analytical stability regions are derived from the corresponding characteristic quasi-polynomials through stability crossing techniques and D-partition methods. Furthermore, optimization-based methodologies are introduced to determine controller gains that improve performance indices associated with spectral abscissa, settling time, and control effort.

Subsequently, the thesis investigates the use of the Proportional Delayed Integral (P δ I) controller for delay-free systems. In this approach, intentional delays are incorporated into the integral action and employed as additional tuning parameters. The obtained results demonstrate that suitable delay selections can significantly improve transient response characteristics and exponential decay rates compared with conventional PI structures.

Finally, delay-based controllers for systems with inherent delays are analyzed through the Integral Retarded (IR) and Integral–Proportional Delayed (I-P δ) controllers. Explicit stability conditions and σ -stability regions are obtained for general classes of delayed plants. The results show that delayed control actions can enlarge admissible stability regions and improve robustness properties while preserving closed-loop stability.

The numerical studies and experimental results support the validity of the proposed approaches, while also highlighting the benefits that can be achieved by deliberately introducing delays into controller structures for different classes of dynamical systems.



Resumen

Esta tesis aborda el análisis, diseño y optimización de controladores basados en retardos para sistemas dinámicos con y sin retardos inherentes. Las metodologías propuestas se desarrollan desde una perspectiva en el dominio de la frecuencia y están sustentadas mediante análisis de cuasipolinomios, fronteras de cruce de estabilidad y procedimientos basados en optimización. El objetivo principal es demostrar que los retardos intencionales pueden incorporarse sistemáticamente en la estructura del controlador para mejorar la estabilidad en lazo cerrado, el desempeño transitorio, la robustez y las propiedades de convergencia exponencial.

En primer lugar, se estudian controladores proporcional–integral (PI) para sistemas con retardos inherentes. Se derivan regiones analíticas de estabilidad a partir de los cuasipolinomios característicos correspondientes mediante técnicas de cruce de estabilidad y métodos de partición-D. Además, se introducen metodologías basadas en optimización para determinar ganancias del controlador que mejoren índices de desempeño asociados con la abscisa espectral, el tiempo de establecimiento y el esfuerzo de control.

Posteriormente, la tesis investiga el uso del controlador Proportional Delayed Integral (P δ I) para sistemas libres de retardo. En este enfoque, se incorporan retardos intencionales en la acción integral y se emplean como parámetros adicionales de ajuste. Los resultados obtenidos demuestran que selecciones adecuadas del retardo pueden mejorar significativamente las características transitorias y las tasas de decaimiento exponencial en comparación con estructuras PI convencionales.

Finalmente, se analizan controladores basados en retardos para sistemas con retardos inherentes mediante los controladores Integral Retarded (IR) e Integral–Proportional Delayed (I-P δ). Se obtienen condiciones explícitas de estabilidad y regiones de estabilidad- σ para clases generales de plantas con retardo. Los resultados muestran que las acciones de control retardadas pueden ampliar las regiones admisibles de estabilidad y mejorar las propiedades de robustez mientras se preserva la estabilidad en lazo cerrado.

Los resultados obtenidos mediante simulaciones numéricas y pruebas experimentales respaldan la validez de las metodologías desarrolladas y evidencian los beneficios de incorporar retardos de manera deliberada en las estructuras de control aplicadas a distintas clases de sistemas dinámicos.

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Quick Summary

What we know is little, and what we are ignorant of is immense.

Pierre-Simon Laplace

In **Chapter 2** introduces the mathematical foundations required throughout the thesis. It presents the basic concepts of linear algebra, differential calculus, Laplace transforms, and linear systems theory. The chapter also introduces complex-variable analysis, characteristic roots, spectral abscissa, exponential stability, and σ -stability. In addition, the fundamental concepts of time-delay systems and quasi-polynomials are discussed, including root continuity, stability crossing curves, and D-partition methods. These tools establish the theoretical framework used for the analysis and design of delay-based controllers in subsequent chapters.

In **Chapter 3** addresses the analysis and optimization of proportional–integral controllers for systems with inherent time delays. Stability regions are characterized analytically through the construction of stability crossing boundaries derived from the corresponding quasi-polynomial characteristic equation. The concepts of spectral abscissa and σ -stability are employed to evaluate the exponential decay properties of the closed-loop system. Furthermore, optimization-based methodologies are introduced to determine controller gains that improve transient performance and robustness while preserving closed-loop stability. The proposed design procedures are subsequently assessed through numerical examples and simulation studies.

In **Chapter 4** investigates the use of intentional delays as additional tuning parameters for the control of delay-free systems. The proposed delay-based controllers are designed to improve transient performance, robustness, and exponential convergence rates without relying on derivative actions. Stability regions are characterized analytically using frequency-domain techniques and stability crossing boundaries. Furthermore, optimization-based methodologies are employed to determine controller parameters that maximize performance indices such as spectral abscissa or settling time. The developed results demonstrate that intentionally introduced delays can significantly enhance closed-loop behavior in a wide class of dynamical systems.

In **Chapter 5** studies the stabilization and performance analysis of time-delay systems using delay-based control structures. First, the Integral Retarded (IR)

controller is analyzed through the characterization of stability crossing boundaries and admissible controller regions. Then, an analytical–metaheuristic framework is introduced to optimize controller parameters according to performance criteria such as maximum exponential decay and control effort. The chapter also introduces the Integral–Proportional Delayed (I-P δ) controller, deriving explicit stability conditions and σ -stability regions for general delayed plants. Several numerical and experimental examples demonstrate the effectiveness of the proposed methodologies.

In **Chapter 6** addresses the modeling and control of a three-degree-of-freedom hybrid robotic fin mechanism inspired by the motion of the manta ray *Mobula alfredi*. The proposed architecture consists of a coupled closed kinematic chain. A Lie-algebra-based formulation is employed to describe the kinematics of the mechanism and to determine the relationships among the active and passive joint coordinates, velocities, and accelerations imposed by the closed-chain constraints. Based on this formulation, the dynamic model of the mechanism is obtained and used to formulate the inverse dynamics problem, allowing the actuator torques required to reproduce prescribed fin trajectories to be determined. Finally, a closed-loop joint-trajectory tracking strategy based on a computed-torque-type control scheme is introduced. The resulting framework provides a systematic connection between the geometric description, constrained dynamics, and control of the robotic fin, establishing the basis for reproducing complex three-dimensional motions inspired by batoid locomotion.

Notation

The following sets, operators, constants and abbreviations are defined according to the standard notation adopted in [1] [2] [3] [4] [5] [6].

Sets

$\mathbb{C}$	Collection of all complex numbers
$\mathbb{C}_+$	Complex numbers whose real component is strictly greater than zero
$\mathbb{C}_-$	Complex numbers whose real component is strictly less than zero
$\mathbb{R}$	Collection of all real numbers
$\mathbb{R}_+$	Real numbers greater than or equal to zero
$\mathbb{R}_-$	Real numbers strictly less than zero
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathbb{Z}$	Collection of all integer numbers
$\mathbb{Z}_+$	Integers greater than or equal to zero
$\mathbb{N}$	Collection of natural numbers
$\mathbb{Q}$	Collection of rational numbers
$\mathbb{PC}$	Class of piecewise-continuous functions
$\Re$	Real axis in the complex plane
$\Im$	Imaginary axis in the complex plane
$\mathcal{C}[a, b]$	Set of continuous mappings $f : [a, b] \rightarrow \mathbb{R}^n$
$\mathcal{C}^m[a, b]$	Set of mappings $f : [a, b] \rightarrow \mathbb{R}^n$ having continuous derivatives up to order m
$\mathcal{A}^*$	Given a set $\mathcal{A}$, $\mathcal{A}^* := \mathcal{A} \setminus \{0\}$
$\overline{\mathcal{A}}$	Given a set $\mathcal{A}$, $\overline{\mathcal{A}} := \mathcal{A} \cup \{0\}$
$\Re(z)$	Real component of z , for any $z \in \mathbb{C}$
$\Im(z)$	Imaginary component of z , for any $z \in \mathbb{C}$

Operators and constants

$i := \sqrt{-1}$	Complex imaginary unit
$\mathcal{L}$	Operator associated with the Laplace transform
$(\vec{x}, \vec{y})$	Inner product of the vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$
$\text{Arg}(z)$	Principal argument associated with $z \in \mathbb{C}$.
$\text{Card}(\mathcal{A})$	Number of elements contained in the set $\mathcal{A}$
$\text{sign}(x)$	For $x \in \mathbb{R}$, $\text{sign}(x) \in \{-1, 0, 1\}$ indicates whether x is negative, zero, or positive
$\ \cdot\ $	Norm induced by the Euclidean metric
$\mathbf{I}_n$	Identity matrix of dimension $n \times n$
$\mathbf{0}_n$	Zero matrix of dimension $n \times n$

Abbreviations

HOT	Higher order terms
LHP	Left-half plane (equivalent to $\mathbb{C}_-$)
RHP	Right-half plane (equivalent to $\mathbb{C}_+$)
LTI	Linear time-invariant
SISO	Single-input single-output
PI	Proportional-Integral (controller)
PD	Proportional-Derivative (controller)
PID	Proportional-Integral-Derivative (controller)
P δ I	Proportional Delayed Integral (controller)
IR	Integral Retarded (controller)
I-P δ	Integral Proportional Delayed (controller)
CRB	Complex Root Boundary
RRB	Real Root Boundary
IRB	Infinite Root Boundary
GA	Genetic Algorithm
MED	Maximum Exponential Decay rate
HTM	Homogeneous Transformation Matrix
CoM	Center of Mass
DoF	Degree of Freedom

Publications

*Time is what prevents everything
from happening at once.*

John Archibald Wheeler

The work carried out in this thesis resulted in the following scientific publications:

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- C2 J.-A. Hernández-Gallardo**, A.-J. Guel-Cortez, E. J. Gonzalez-Galvan & C.-F. Méndez-Barrios, *Designing Proportional Delayed Integral Control for Fast Regulation in Second-Order Systems: A Geometric Approach*, 2023 9th International Conference on Control, Decision and Information Technologies (CoDIT), Rome, Italy, 2023, pp. 01-06. [\[8\]](#)
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- C4 J.-A. Hernández-Gallardo** et al., *A Synergistic Approach to Velocity Control under Intentional and Inherent Time Delays*, 2025 11th International Conference on Control, Decision and Information Technologies (CoDIT), Split, Croatia, 2025, pp. 297-302. [\[10\]](#)
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- B2 Hernández-Gallardo, J.-A.**, Torres-García, D., Moreno-Negrete, E., Garcia Cena, C.-E., Gonzalez-Galvan, E. J. & Méndez-Barrios, C.-F. (2026). *Delay-Based State Feedback Control for the Inertial Wheel Pendulum*. In: Tafur, J., Baca, J., Moreno, H.A., Cardona, M.N., Calderón Chavarri, J.A. (eds) *Advances in Automation and Robotics Research*. LACAR 2025. *Lecture Notes in Networks and Systems*, vol 1853. Springer, Cham. [13]

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- J1 J.-A. Hernández-Gallardo**, C.-F. Méndez-Barrios, and A.-J. Guel-Cortez, *A synergistic approach to proportional-integral controller design in linear time-delay systems*, *Asian J. Control*, (2025), 1–15. [14]
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- J3 Hernández-Gallardo, J.-A.**, Moreno-Negrete, Méndez-Barrios, C.-F. *A Integral-Retarded Control Strategy for Time-Delay Systems: A Synergistic Approach*. *Int. J. Dynam. Control*. (Under review)
- J4 Hernández-Gallardo, J.-A.**, Méndez-Barrios, C.-F. *Synergistic Optimization of Delay-Based Controllers for Trajectory Tracking in Tilt-Rotor UAVs*. (To be submitted)
- J5 Hernández-Gallardo, J.-A.**, Méndez-Barrios, C.-F., Moreno-Negrete, E. et al. *On the effectiveness of proportional delayed integral controller for linear time-invariant applications*. (To be submitted)

Collaborations

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- C9** A. Duman Mammadov, E. Moreno-Negrete, **J.-A. Hernández-Gallardo**, E. Dincel, C.-F. Méndez-Barrios, & M. T. Söylemez, *Digital Design of Delay-Based Control for Hybrid Switch-Mode DC-DC Converters*, 23rd IFAC World Congress, Busan, Republic of Korea, 2026. (To be published).
- J6** Torres-García, D., **Hernández-Gallardo, J.-A.**, Méndez-Barrios, C.-F. *PI Controller Design for σ -Stabilization of Linear μ -Order Systems with Input Delay*. (To be submitted)

1 Introduction

Simplicity is the ultimate sophistication.

Leonardo da Vinci

Contents of this chapter

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1.1 Problem Formulation

In recent decades, the control of nonlinear and multivariable dynamical systems has become a central topic in several engineering fields, particularly in robotics, aerospace systems, and power electronics. Platforms such as serial and parallel manipulators, bioinspired robots, tilt-rotor unmanned aerial vehicles, and DC–DC power converters share complex dynamic characteristics, including nonlinearities, internal coupling, parameter dependence, and sensitivity to external disturbances. These features significantly complicate the design of control strategies capable of ensuring stability, dynamic performance, and robustness simultaneously.

In robotics and aerospace applications, these challenges arise in trajectory tracking, disturbance rejection, and transient response behavior, especially in systems with multiple degrees of freedom or varying operating regimes. Similarly, in power electronics, DC–DC converters exhibit fast dynamics, nonlinear switched behavior, and strong sensitivity to load variations, electrical parameter uncertainties, and supply disturbances, which demand control laws capable of guaranteeing stable and accurate regulation under changing conditions.

Despite these complexities, classical proportional–integral, proportional–derivative and proportional–integral–derivative controllers remain widely used in both industrial and experimental settings due to their simple structure and ease of implementation [20] [21]. However, their performance often degrades when applied to high-order systems, systems with nonminimum-phase behavior, or nonlinear dynamics with strict transient performance requirements. Moreover, traditional tuning methods do not provide systematic criteria to guarantee robust

stability or to explicitly optimize the trade-off between response speed, damping, and disturbance sensitivity.

Within this context, delay-based controllers have emerged as a promising alternative to enhance system performance, even in systems without inherent physical delays. The intentional introduction of delays in the control law has been shown to favorably modify the closed-loop spectral structure, leading to improved damping, reduced overshoot, and faster transient responses [22] [23]. These advantages have been reported in mechanical and aerospace systems, as well as in power electronics applications, including DC–DC converters operating under varying load conditions.

Nevertheless, the inclusion of delays introduces significant analytical challenges, since the resulting dynamics belong to the class of time-delay systems characterized by functional differential equations with infinitely many characteristic roots. Stability depends on complex and often non-convex regions in the parameter space, which complicates systematic controller design. Although notable advances exist in the literature, most approaches focus on limited stability analyses or specific system configurations, without providing a general framework applicable to nonlinear systems of practical interest or diverse experimental platforms.

Furthermore, tuning strategies for delay-based controllers often treat the delay as a fixed or empirically adjusted parameter, rather than integrating it within an explicit optimization framework that balances stability and performance. This limitation is particularly critical in experimental applications such as DC–DC converters and tilt-rotor vehicles, where small parameter variations can significantly affect closed-loop behavior.

As a result, there remains a lack of a unified methodological framework that integrates delay-based control, rigorous spectral stability analysis, and parameter optimization, applicable to both mechanical and aerospace systems as well as power electronic systems. This gap limits the transfer of theoretical results into reliable and reproducible experimental implementations and restricts the generalization of observed performance improvements across different application domains.

Accordingly, a systematic control framework is required in which delays are deliberately embedded in the control law and their influence is examined through rigorous stability analysis. The resulting design should also incorporate coordinated parameter optimization to achieve suitable closed-loop performance. Moreover, the proposed framework must be sufficiently general to address complex nonlinear systems and should be verified through analytical developments, numerical studies, and experimental tests on physical platforms, such as DC–DC power converters and robotic or aerospace systems.

Why replace the derivative operator with the delay operator?

In a conventional PID controller, the derivative term is generally employed to accelerate the transient dynamics and provide additional damping to the closed-loop response. However, its practical implementation may be problematic because the ideal derivative operator exhibits an unbounded gain as the frequency increases. Consequently, high-frequency components, measurement noise, and small signal variations can be significantly amplified [5] [24].

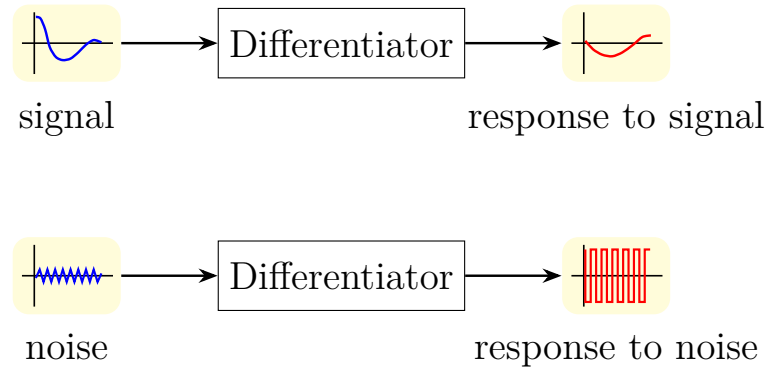


Figure 1.1 / Noise problem in derivative action.

A possible alternative is to replace the explicit derivative action by a retarded term constructed from the current and past values of the tracking error. Starting

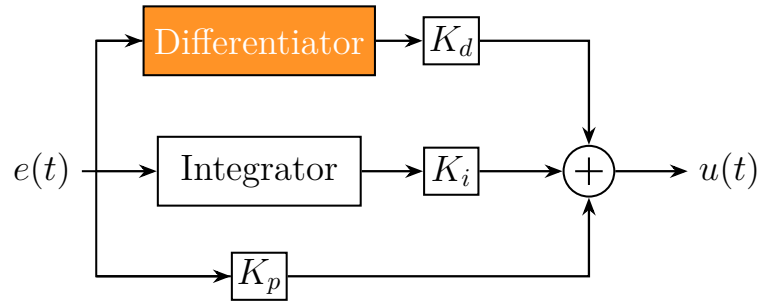


Figure 1.2 / Block diagram representation of the PID control structure.

from the conventional PID control law,

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \dot{e}(t), \quad (1.1)$$

the derivative of the error can be approximated by means of the backward Euler formula as

$$\dot{e}(t) \approx \frac{e(t) - e(t-h)}{h}, \quad (1.2)$$

where $h > 0$ denotes the time interval between the current and delayed values of the error signal. Substituting (1.2) into (1.1) yields

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{e(t) - e(t-h)}{h}.$$

Expanding the last term gives and, after grouping the terms associated with the current error,

$$u(t) = \left(K_p + \frac{K_d}{h}\right)e(t) + K_i \int_0^t e(\tau) d\tau - \frac{k_d}{h} e(t-h).$$

Defining the new controller gains as

$$k_p = K_p + \frac{K_d}{h}, \quad k_i = K_i, \quad k_r = \frac{K_d}{h},$$

the control law can be rewritten as

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau - k_r e(t-h), \quad (1.3)$$

which corresponds to a proportional–integral–retarded (PIR) controller.

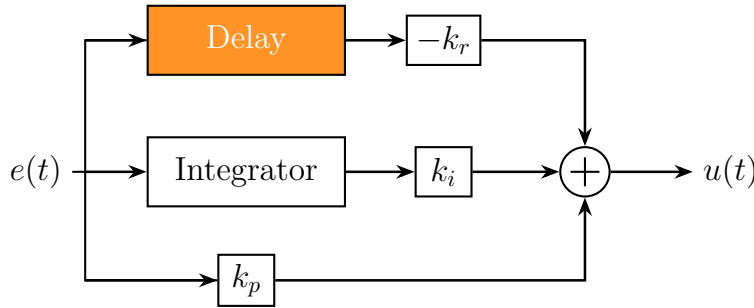


Figure 1.3 / PIR controller diagram.

Therefore, the derivative action of the conventional PID structure can be interpreted, through a finite-difference approximation, as the combination of a proportional contribution associated with the current error and a retarded contribution associated with its past value.

From an operator perspective, the approximation in (1.2) can be written in the Laplace domain as

$$s \approx \frac{1 - e^{-hs}}{h}. \quad (1.4)$$

This expression provides an additional motivation for introducing the retarded term. In particular, for $s = j\omega$,

$$\frac{1 - e^{-j\omega h}}{h} = e^{-j\omega h/2} \frac{2j}{h} \sin\left(\frac{\omega h}{2}\right). \quad (1.5)$$

For sufficiently small values of ωh , the approximation

$$\sin\left(\frac{\omega h}{2}\right) \approx \frac{\omega h}{2} \quad (1.6)$$

leads to

$$\frac{1 - e^{-j\omega h}}{h} \approx j\omega e^{-j\omega h/2}. \quad (1.7)$$

Thus, the retarded operator exhibits a derivative-like behavior within the low-frequency range. Nevertheless, an important difference appears at high frequencies. While the magnitude of the ideal derivative operator satisfies

$$|j\omega| = \omega, \quad (1.8)$$

and therefore grows without bound as $\omega \rightarrow \infty$, the magnitude of the retarded operator is bounded according to

$$\left| \frac{1 - e^{-j\omega h}}{h} \right| = \frac{2}{h} \left| \sin\left(\frac{\omega h}{2}\right) \right| \leq \frac{2}{h}. \quad (1.9)$$

Consequently, replacing the derivative term by a retarded action provides a controller structure that retains a derivative-like contribution over a certain frequency range while avoiding the unbounded high-frequency gain of the ideal derivative operator. Moreover, once the PIR structure is adopted, the delay h can be regarded not only as a parameter arising from the finite-difference approximation, but also as an intentional design parameter that can be selected together with the controller gains to shape the closed-loop stability and dynamic response.

1.2 Objectives

General objective

Design and implement delay-based control strategies that exponentially stabilize systems described by functional differential equations, while improving the performance of their closed-loop dynamic response to disturbances and uncertainties.

Specific objectives

- Study and analyze delay-based reduced-order PID controllers in order to understand the effect of intentionally introduced delay on the dynamics of the closed-loop system and its influence on the spectral location of the dominant poles.
- Determine analytical criteria that allow controllers to be designed in the parameters k_p , k_i , k_r , h depending on the controller used, which guarantee maximum exponential decay of the system, maximizing the value of the

negative spectral abscissa ($\sigma_{\max}$) as a direct metric of dynamic performance and response speed.

- Establish formal criteria for obtaining critical delay (h_c) in delay-based controllers, identifying stability boundaries in parameter space and analyzing the transition between stability and loss of stability in systems with delay.
- Ensure the robustness of the designed controllers against uncertainties in the delay value, evaluating the sensitivity of the system to parametric variations and proposing criteria to avoid fragile configurations in terms of stability and performance.
- Select delay-based controllers that are non-fragile and optimal, using a combination of spectral analysis and optimization schemes, so as to ensure an appropriate compromise between robustness, stability, and dynamic performance.
- Experimentally validate the theories and methodologies developed throughout the work, using real available platforms, such as CD–CD converters and mechanical dynamic systems, in order to demonstrate the practical feasibility of the proposed approach.

1.3 Hypothesis

The objectives outlined in the previous subsection are connected to the following research hypothesis.

Hypothesis 1.1

The intentional and structured introduction of delays in low-order PID controllers, designed using spectral criteria and optimized in the parameter space, allows for systematic improvement of the dynamic performance and robustness of the closed-loop system, ensuring maximum exponential stability and reducing fragility in the presence of uncertainties, compared to classical controllers without delay, both in mechanical and in power electronics applications.

Starting from this central hypothesis, the following derived hypotheses are established to guide the theoretical development, controller design, and numerical and experimental validation of the available systems.

- H1** The incorporation of an appropriately selected delay in the control law allows the spectral abscissa of the closed-loop system to be shifted favorably, increasing the maximum exponential decay rate ($\sigma_{\max}$) and improving dynamic performance metrics such as settling time and damping.
- H2** There exists a critical delay value (h_c) that defines the stability boundary of delay-based controllers, which can be determined analytically through

spectral analysis, enabling an accurate characterization of stability regions in the parameter space (k_p, k_i, h) .

- H3** Delay-based controllers designed by explicitly considering spectral sensitivity to delay variations exhibit greater robustness to parametric uncertainties than empirically tuned controllers, thereby reducing closed-loop fragility.
- H4** Formulating the delay-based controller design as a spectral optimization problem enables the identification of non-fragile and optimal configurations, achieving an appropriate trade-off between stability, dynamic performance, and robustness to disturbances and parameter variations.
- H5** The proposed delay-based control framework, supported by spectral criteria and parameter optimization, is consistently applicable across different application domains, including mechanical systems and DC–DC power converters, yielding observable performance improvements in both simulations and experimental results.
- H6** The theoretical and numerical improvements obtained through the proposed approach are reflected in reproducible experimental results, demonstrating that delay-based controllers can be implemented on real platforms without compromising system stability or performance.

1.4 State of the Art

Proportional-Integral-Derivative (PID) controllers continue to play a central role in industrial automation and control systems due to their simplicity, reliability, and ease of implementation [4] [25] [26]. In fact, PID-based strategies remain one of the most widely adopted control methodologies in industrial applications, being employed in more than 90% of industrial processes [27]. Their success is mainly attributed to the clear physical interpretation of their parameters and their capability to provide satisfactory closed-loop performance under a broad range of operating conditions.

Among PID variants, the Proportional-Integral (PI) controller is arguably the most commonly used configuration in practice. This preference emerged after recognizing several drawbacks associated with the derivative action, particularly its sensitivity to high-frequency measurement noise [20] [25]. Since the derivative term amplifies noise, its implementation may produce excessive control effort, actuator wear, and even closed-loop instability. Consequently, many practical systems avoid derivative action or incorporate additional filtering stages, which increase controller complexity and may introduce undesirable phase lag.

To alleviate these issues, several alternative controller architectures have been proposed, including I-PD and PI-D configurations [28] [29]. These approaches relocate the derivative action to the feedback path in order to reduce derivative kick during reference changes. Nevertheless, the fundamental issue associated

with noise amplification remains unresolved.

In recent years, delay-based controllers (DBC) have emerged as an attractive alternative for improving the performance of low-order controllers [8] [9] [30] [31]. Contrary to the traditional perspective in which delays are regarded exclusively as destabilizing effects, DBCs intentionally incorporate delays as additional design parameters capable of enhancing transient performance and robustness properties.

One of the earliest approaches in this direction can be found in [32], where a proportional retarded controller was proposed for stabilizing oscillatory systems. Later, several works explored delayed versions of classical controllers, including proportional-delayed (PD-like) structures [30] [33] and proportional-integral-delay formulations [34] [35]. These studies demonstrated that intentionally introduced delays can approximate derivative effects while inherently filtering high-frequency noise.

Within this framework, the P δ I controller introduced in [35] represents a significant advancement. In this controller, the integral action is modified through the inclusion of a delayed error term, allowing improvements in transient response while preserving zero steady-state error. However, despite the promising results reported in the literature, previous works do not provide systematic tuning methodologies for selecting the delay parameter.

Similarly, the Integral-Retarded (IR) controller proposed in [36] incorporates delayed feedback to preserve the benefits of derivative-like action without explicitly using derivatives. This controller guarantees zero steady-state tracking error while reducing sensitivity to measurement noise. More recently, several studies have explored the application of delay-based structures in practical systems, including robotic platforms, electromechanical systems, and DC motor control applications.

The stability analysis of delay-based controllers has been strongly supported by geometric methods in the controller parameter space. In particular, the $\mathcal{D}$ -decomposition method [37] has become a fundamental tool for characterizing stability regions in systems with delays. The method, first proposed by Neimark in the 1940s, divides the parameter domain into subregions characterized by the same number of unstable roots. These subregions are delimited by crossing boundaries associated with roots passing through the imaginary axis.

The $\mathcal{D}$ -decomposition approach has been successfully employed for the design of PI, PD, and delay-based controllers [34] [35]. Furthermore, combining $\mathcal{D}$ -decomposition with spectral abscissa analysis [38] enables the computation of σ -stability regions and maximum exponential decay rates, allowing the design of controllers with optimal convergence properties.

Another important challenge in control design arises in non-minimum phase systems (NMPS). These systems appear in a wide variety of engineering applications, including DC-DC converters, hydraulic turbines, aircraft dynamics, and biochemi-

cal reactors [39]. The existence of unstable zeros imposes severe performance limitations, constraining achievable bandwidth and generating inverse responses.

Although several methodologies have been proposed for controlling NMPS [40] [41] [42], most existing approaches focus primarily on stable plants and often neglect the complexities introduced by unstable non-minimum phase behavior. Consequently, there remains a need for alternative control methodologies capable of systematically addressing these difficulties while preserving robustness and implementation simplicity.

In parallel, optimization techniques based on metaheuristic algorithms have gained considerable attention in modern controller tuning [43] [44] [45]. Approaches based on genetic algorithms, particle swarm optimization, and other evolutionary methods have demonstrated their effectiveness in solving multi-objective optimization problems involving robustness, fragility reduction, and performance improvement. These methods have also been combined with delay-based controller design to obtain improved closed-loop responses [12] [14].

Motivated by the previous discussion, this work focuses on the analysis, tuning, and optimization of delay-based low-order controllers for linear systems, including systems with delays and non-minimum phase behavior. The proposed methodology combines geometric stability analysis, $\mathcal{D}$ -decomposition techniques, and performance-oriented optimization criteria in order to obtain controllers with improved transient response, robustness, and practical implementability.

Mathematical Preliminaries

Mathematical analysis is as extensive as nature itself.

Joseph Fourier

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This chapter introduces the mathematical foundations required throughout the thesis. It presents the basic concepts of linear algebra, differential calculus, Laplace transforms, and linear systems theory. The chapter also introduces complex-variable analysis, characteristic roots, spectral abscissa, exponential stability, and σ -stability. In addition, the fundamental concepts of time-delay systems and quasi-polynomials are discussed, including root continuity, stability crossing curves, and D-partition methods. These tools establish the theoretical framework used for the analysis and design of delay-based controllers in subsequent chapters.

2.1 Types of Systems

Consider a dynamical system whose evolution is described by a finite collection of coupled first-order ordinary differential equations. In compact notation, the corresponding first-order vector model can be written as

$$\dot{x} = f(x, u), \quad (2.1)$$

where $x \in \mathbb{R}^n$ denotes the state vector and $u \in \mathbb{R}^p$ represents the control input. When required, an output equation may also be incorporated to describe the measured or controlled variables, as shown below

$$u = h(x, u), \quad (2.2)$$

where $x \in \mathbb{R}^m$ denotes the *output vector*. Often, the input u does not appear explicitly in (2.1), either because the input is zero or because it was specified as a

function of the state $u = \gamma(x)$ —feedback control. In this case, the state equation is the unforced equation

$$\dot{x} = f(x), \quad (2.3)$$

Stationary or time-invariant systems are so named because their behavior remains unchanged when the time origin is shifted. When the system is nonstationary or time-variant, the right-hand side of (2.3) is an explicit function of time.

DEFINITION

2.1

Equilibrium Points [46]. A state $x = x^*$ is said to be an *equilibrium point* (EP) of (2.3) whenever the system, once initialized at x^* , does not evolve away from that state. Equivalently, if $x(0) = x^*$, then $x(t) = x^*$ for every subsequent instant of time.

Remark 2.1. The EP in (2.3) are the roots of the equation $f(x) = 0$ ┘

Remark 2.2. An EP may be *isolated*, meaning that there are no other equilibrium points in its neighborhood, or there may exist a *continuum* of equilibrium points. When the system is linear, (2.3) takes the familiar form

$$\dot{x} = \mathbf{A}x + \mathbf{B}u, \quad (2.4)$$

where $\mathbf{A}$ denotes the system matrix and $\mathbf{B}$ corresponds to the input matrix. For the zero-input case, $u = 0$, the origin $x = 0$ is the only isolated equilibrium point. ┘

Since the analysis and control of linear systems are well established, it is generally convenient to begin the study of a nonlinear system by linearizing it about an equilibrium point and analyzing the resulting linear model. However, linearization only predicts the local behavior of the system and is therefore inadequate for describing its dynamics far from the operating point.

A continuous-time linear-time-invariant (LTI) system can also be represented as

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad (2.5a)$$

$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t), \quad (2.5b)$$

where

$$x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad y(t) \in \mathbb{R}^p.$$

The matrices

$$\mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n}, \quad \mathbf{D} \in \mathbb{R}^{p \times m}$$

are called the system, input, output, and feedthrough matrices, respectively.

To determine whether a system can be controlled through its inputs and whether its internal state can be determined through its outputs, the concepts of controllability and observability are introduced.

DEFINITION
2.2

Controllability [47]. The state-space pair $(\mathbf{A}, \mathbf{B})$ is controllable when its controllability matrix

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \quad (2.6)$$

has maximum rank. Thus, controllability is guaranteed if

$$\text{rank}(\mathcal{C}) = n. \quad (2.7)$$

DEFINITION
2.3

Observability [47]. The state-space pair $(\mathbf{A}, \mathbf{C})$ is observable whenever the corresponding observability matrix

$$\mathcal{O} = [\mathbf{C}^\top \quad (\mathbf{CA})^\top \quad (\mathbf{CA}^2)^\top \quad \cdots \quad (\mathbf{CA}^{n-1})^\top]^\top \quad (2.8)$$

has full rank. Equivalently, observability requires

$$\text{rank}(\mathcal{O}) = n. \quad (2.9)$$

Remark 2.3. Controllability and observability determine whether the internal state of a linear system can be fully manipulated through the input and reconstructed from the output. $\lrcorner$

DEFINITION
2.4

Hurwitz matrix [47]. A matrix $\mathbf{A}$ is called *Hurwitz* if all its eigenvalues satisfy

$$\Re(\lambda_i) < 0.$$

Consider the following system:

$$\dot{x}(t) = f(t, x(t), x(t-h)), \quad (2.10)$$

where $h \geq 0$, $t \geq t_0$, $t_0 \in \mathbb{R}$, and $f(t, x, y)$ is a well-defined function of its arguments. Furthermore, it is assumed that f is continuous with respect to all of its arguments.

DEFINITION
2.5

Initial Condition. Given $t_0 \in \mathbb{R}$, the *initial value problem* associated with (2.10) is defined by specifying an initial time t_0 together with an initial function φ . A solution $x(t)$ is then required to satisfy

$$x(\theta) = \varphi(\theta), \quad \forall \theta \in [t_0 - h, t_0], \quad (2.11)$$

where φ belongs to an appropriate functional space, such as $\mathcal{C}([t_0 - h, t_0], \mathbb{R}^n)$ or $\mathbb{PC}([t_0 - h, t_0], \mathbb{R}^n)$.

DEFINITION

2.6

State. Given an initial time $t_0 \in \mathbb{R}$ and an instant $t_1 \geq t_0$, the *state* of a dynamical system is defined as the minimum amount of information required to uniquely determine the future evolution of the system for all $t \geq t_1$.

DEFINITION

2.7

Delay-Free Systems. Consider the *delay-free dynamical system*

$$\dot{x}(t) = f(t, x(t)), \quad (t_0, x_0), \quad (2.12)$$

where x_0 is the initial condition. Since the state $x(t)$ is available instantaneously at any time $t \geq t_0$, the future evolution of the system can be determined solely from the current state. Therefore, to continue the system dynamics for $t \geq t_0$, it is sufficient to know

$$x_0 := x(t_0).$$

Time-delay systems are dynamical systems whose evolution depends on past values of the state, input, or output. Such delays arise from sensing, communication, computation, actuation, filtering, or transport phenomena.

A delayed signal is written as

$$x(t - h), \quad h > 0.$$

Different classifications exist for time-delay systems. Among the most common are the following:^{1,2}

(i) **Retarded-type DDE**

$$\dot{\mathbf{x}}(t) = a\dot{\mathbf{x}}(t - h) + b\mathbf{x}(t) \quad (2.13)$$

(ii) **Neutral-type DDE**

$$\ddot{\mathbf{x}}(t) = a\ddot{\mathbf{x}}(t - h) + b\mathbf{x}(t - \tau) \quad (2.14)$$

(iii) **Advanced-type DDE**

$$\dot{\mathbf{x}}(t - h) = \mathbf{x}(t) \quad (2.15)$$

(iv) **DDEs with distributed delays**

$$\dot{\mathbf{x}}(t) = \int_{-h}^0 f(t, \mathbf{x}(y + \theta)) d\theta \quad (2.16)$$

¹ODE: Ordinary Differential Equations.

²DDE: Delay Differential Equations.

(v) DDEs with time-dependent delays

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{x}(t - \tau(t))) \quad (2.17)$$

This work is focused on retarded-type systems.

DEFINITION

2.8

A time-delay system is said to be of *retarded type* if delays do not appear in the highest-order derivative terms. It is said to be of *neutral type* if delayed terms appear in the highest-order derivatives.

Remark 2.4. The thesis focuses on retarded-type quasi-polynomials, for which the continuity of characteristic roots with respect to parameters provides the basis for stability crossing methods. ┘

Remark 2.5. Retarded systems possess infinitely many characteristic roots with only a finite number in any right half-plane. ┘

It is important to note that, in the theory of time-delay systems, the initial state belongs to a functional space. Consequently, the initial condition must be defined through an initial function $\phi : [-h, 0] \rightarrow \mathbb{R}^n$, where h denotes the delay associated with the system. In this context, [23] presents the initial value problem for a system of the form:

$$\frac{dx(t)}{dt} = g(t, x(t), x(t - h)), \quad (2.18)$$

where an initial instant $t_0 \geq 0$ and an initial function φ are considered. The objective is to determine a solution satisfying the initial condition

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-h, 0]. \quad (2.19)$$

In this framework, the initial function is commonly assumed to be continuous, although piecewise continuous functions may also be considered depending on the problem under study. It is worth noting that the particular case defined by

$$t_0 = 0, \quad \varphi(t) = 0, \quad \forall t \in [-h, 0],$$

is admissible since $\varphi = 0$ belongs to the space of continuous functions. This particular case is referred to as the *trivial initial condition*. Throughout this work, the trivial initial condition will be denoted by 0_h .

Figure 2.1 illustrates how a solution is constructed from its past values.

The use of an initial condition defined over an interval, as described above, leads time-delay systems to be regarded as infinite-dimensional systems [23]. The following definition helps clarify this concept.

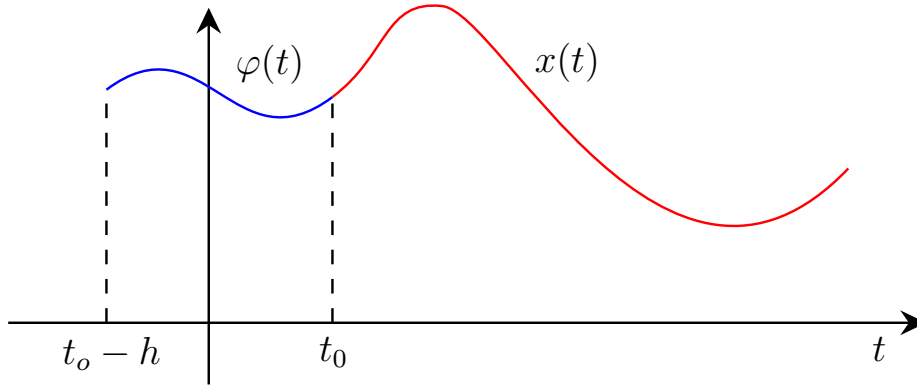


Figure 2.1 / Time-delay system and its initial condition.

2.2 Stability

To introduce the following concepts, it is first necessary to specify the properties of the differential equation describing the dynamical system. Consider the nonlinear system

$$\dot{x}(t) = f(t, x), \quad (2.20)$$

and let $\varphi(t)$ be a solution of (2.20) defined on $[0, +\infty)$. Throughout this section, it is assumed that there exists a constant $a > 0$ such that

$$f \in \mathcal{C}^1(\Delta_a, \mathbb{R}^n), \quad (2.21)$$

where

$$\Delta_a = \{(t, x) : t \geq 0, \|x(t) - \varphi(t)\| < a\}. \quad (2.22)$$

The concept of stability is one of the fundamental criteria for evaluating the dynamic behavior of a system, since it determines whether the state variables and the output remain bounded or converge to an equilibrium point following a disturbance or in response to an external input.

Depending on the context of analysis and the design objectives, there are different notions of stability that describe specific properties of the system's response. Among the most relevant are Lyapunov stability, asymptotic stability, exponential stability, and input-output (BIBO) stability, each with its own mathematical criteria and specific applications.

DEFINITION 2.9

Stability [48]. A solution $\varphi(t)$ is said to be *stable* on $[t_0, +\infty)$ if, for every $\epsilon > 0$ and every $t_0 \geq 0$, there exists a constant $\delta = \delta(\epsilon, t_0) > 0$ such that, for every initial condition $x_0 \in B_\delta(\varphi(t_0))$, the corresponding solution

$$x(t; t_0, x_0) \quad (2.23)$$

is defined on $[t_0, +\infty)$ and satisfies

$$\sup_{t \geq t_0} \|x(t; t_0, x_0) - \varphi(t)\| < \epsilon. \quad (2.24)$$

The graphical interpretation of the previous definition is shown in Figure 2.2.

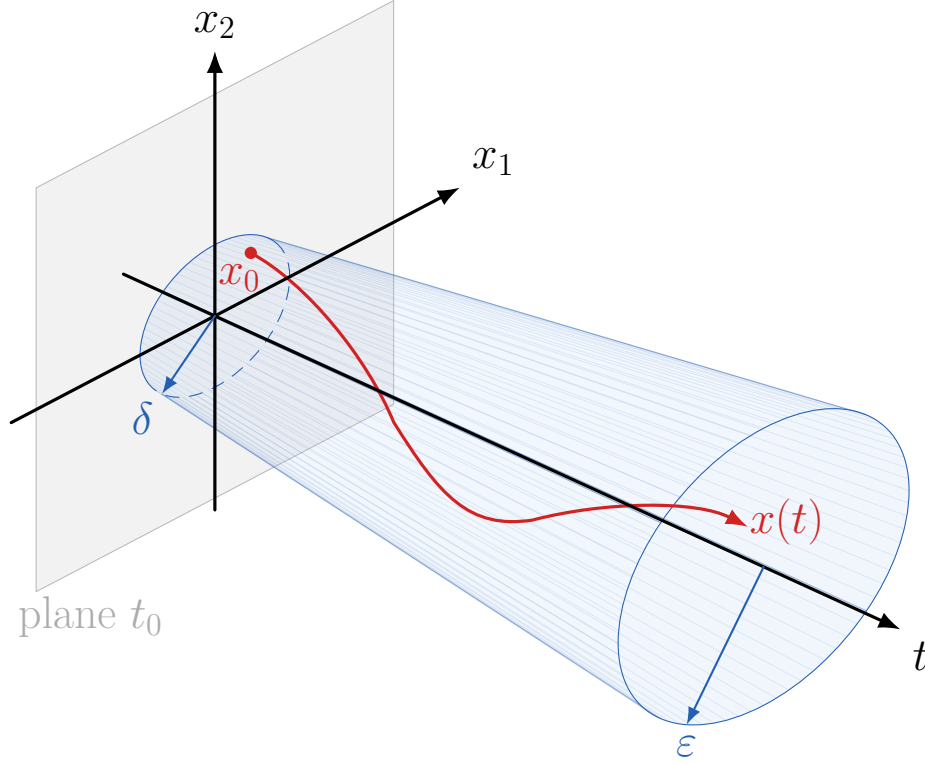


Figure 2.2 / Stability.

DEFINITION 2.10

Asymptotic Stability [48]. A solution $\varphi(t)$ is said to be *asymptotically stable* on $[t_0, +\infty)$ if it is stable and, in addition, for every $t_0 \geq 0$, there exists a constant $\gamma = \gamma(t_0) > 0$ such that, for every initial condition

$$x_0 \in B_\gamma(\varphi(t_0)),$$

the corresponding solution satisfies

$$\lim_{t \rightarrow \infty} \|x(t; t_0, x_0) - \varphi(t)\| = 0. \quad (2.25)$$

Similarly, the definition above can be better understood with the diagram shown in Figure 2.3.

For linear systems, these characteristics are commonly assessed through the position of the system poles in the complex plane. In contrast, nonlinear and time-

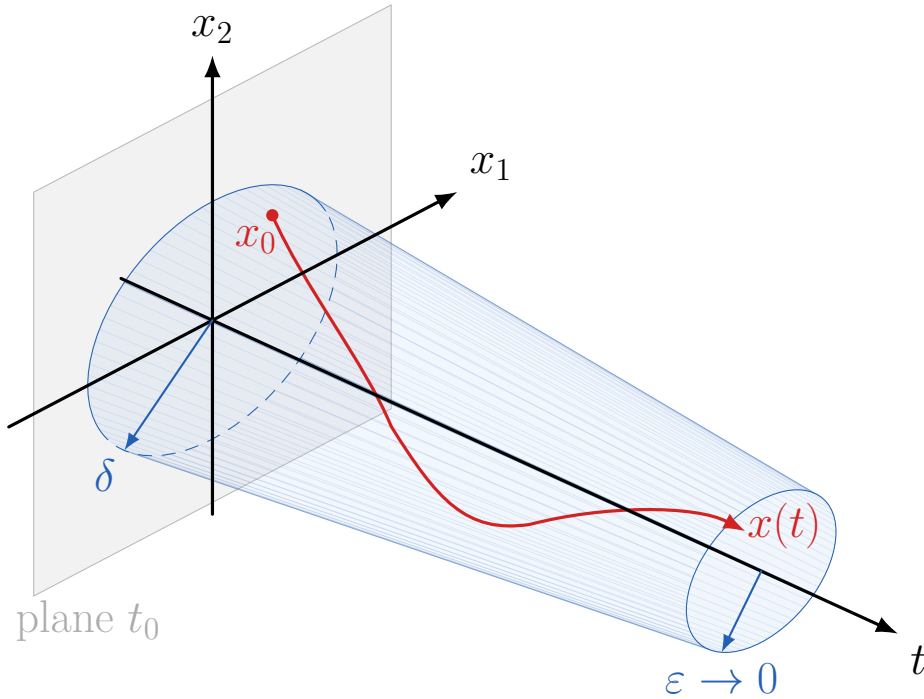


Figure 2.3 / Asymptotic stability.

delay systems generally require broader analytical tools for their evaluation. The study of these concepts provides the theoretical foundation for the analysis and design of controllers capable of ensuring safe, robust behavior with the required dynamic performance.

Following the previous definitions, the following result regarding linear systems is stated.

THEOREM
2.1

Asymptotic stability in linear systems. The equilibrium point $\mathbf{x} = 0$ of

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

is *asymptotically stable* iff $\mathbf{A}$ is Hurwitz.

2.3 Complex Variables

Complex-variable concepts are essential for describing poles, characteristic roots, frequency-domain behavior, and stability boundaries.

A complex number is written as

$$z = \sigma + i\omega,$$

where $\sigma, \omega \in \mathbb{R}$. Its real and imaginary parts are denoted by

$$\Re(z) = \sigma, \quad \Im(z) = \omega,$$

and its modulus is

$$|z| = \sqrt{\sigma^2 + \omega^2}.$$

The open left-half plane and open right-half plane are defined as

$$\mathbb{C}_- = \{z \in \mathbb{C} : \Re(z) < 0\}, \quad \mathbb{C}_+ = \{z \in \mathbb{C} : \Re(z) > 0\}.$$

Euler's formula,

$$e^{i\theta} = \cos(\theta) + i \sin(\theta),$$

connects complex exponentials with oscillatory signals. Therefore, a root $s = \sigma + i\omega$ produces the modal response

$$e^{st} = e^{\sigma t} e^{i\omega t},$$

where σ determines decay or growth and ω determines oscillation.

Applying the Laplace transform to (2.5) under zero initial conditions gives

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \quad (2.26)$$

Proof. Taking the Laplace transform of (2.5) under zero initial conditions,

$$sX(s) = \mathbf{A}X(s) + \mathbf{B}U(s), \quad Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s).$$

Hence,

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s),$$

and therefore

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s).$$

Substituting this expression into the output equation yields

$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s),$$

from which the result follows. $\square$

DEFINITION 2.11

Transfer function. Consider a continuous-time LTI system described by

$$a_n y^{(n)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t). \quad (2.27)$$

Assuming zero initial conditions, the *transfer function* is defined as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \quad (2.28)$$

where $Y(s)$ and $U(s)$ are the Laplace transforms of the output and input, respectively, and $a_i, b_j \in \mathbb{R}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

The polynomial

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

is called the characteristic polynomial, whereas

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

is called the characteristic equation.

Remark 2.6. The poles of the transfer function correspond to the roots of the denominator and determine the stability properties of the system. The zeros correspond to the roots of the numerator and influence the system dynamics. ─

DEFINITION
2.12

Causality. A rational transfer function $G(s) = \frac{N(s)}{D(s)}$ is said to be proper if

$$\deg(D(s)) \geq \deg(N(s)),$$

and strictly proper if

$$\deg(D(s)) > \deg(N(s)).$$

Properness is associated with causality in finite-dimensional LTI systems.

DEFINITION
2.13

Characteristic root. Let $\Delta(s)$ be a characteristic function. Any solution $s_i \in \mathbb{C}$ of $\Delta(s_i) = 0$ is called a *characteristic root*.

Another way to define the asymptotic stability of a linear system, but in the frequency domain, is shown below.

DEFINITION
2.14

Asymptotic stability. A continuous-time linear system is *asymptotically stable* if all its characteristic roots satisfy

$$\Re(s_i) < 0.$$

DEFINITION
2.15

Finite and Infinite dimensional system. A dynamical system is said to be *finite-dimensional* if the determination of a particular solution requires only a finite amount of information. In this case, the cardinality of the required information set defines the dimension of the system. Conversely, if the amount of required information is infinite, then the system is referred to as *infinite-dimensional*.

It is important to emphasize that the dimension of a system is independent of whether the system is linear or nonlinear. Consequently, a system may be infinite-dimensional regardless of its nature. When the system is linear, frequency-domain tools can be employed for its analysis, making the Laplace transform particularly useful. The following sections study the behavior of the solutions of this class of systems from a frequency-domain perspective.

DEFINITION
2.16

Delay differential equation. A *delay differential equation* is a differential equation in which the derivative of the state depends on delayed values of the state or input.

Remark 2.7. A linear example from above definition is

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0 \mathbf{x}(t) + \mathbf{A}_1 \mathbf{x}(t - h).$$

┘

Unlike ordinary differential equations, delay differential equations require an initial function

$$\phi : [-h, 0] \rightarrow \mathbb{R}^n, \quad \mathbf{x}(t) = \phi(t), \quad t \in [-h, 0].$$

In the Laplace domain, a delayed transfer function is commonly written as

$$G(s) = \frac{P(s)}{Q(s)} e^{-hs},$$

where $P(s)$ and $Q(s)$ are polynomials and $h > 0$ is the delay.

DEFINITION
2.17

Quasi-polynomial. A *quasi-polynomial* is a characteristic function of the form

$$\Delta(s) = P_0(s) + \sum_{k=1}^m P_k(s) e^{-h_k s},$$

where $P_k(s)$ are polynomials and $h_k > 0$ are delay parameters.

Remark 2.8. Unlike ordinary polynomials, quasi-polynomials generally possess infinitely many characteristic roots. This property makes time-delay systems infinite-dimensional.

┘

DEFINITION
2.18

Exponential Stability in Laplace domain. Let $h > 0$ and consider the linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0 \mathbf{x}(t) + \sum_{j=1}^r \mathbf{A}_j \mathbf{x}(t - jh). \quad (2.29)$$

The trivial solution $x_0(t; t_0, 0_h)$ is said to be *exponentially stable* if there exist constants $\Delta_0 > 0$, $\sigma > 0$, and $\gamma \geq 1$ such that, for every $t_0 \geq 0$ and every initial function

$$\varphi \in \mathbb{PC}([t_0 - h, t_0], \mathbb{R}^n),$$

satisfying

$$\|\varphi\|_h < \Delta_0,$$

the following inequality holds:

$$\|x(t; t_0, \varphi)\| \leq \gamma \|\varphi\|_h e^{-\sigma(t-t_0)}, \quad \forall t \geq t_0. \quad (2.30)$$

DEFINITION
2.19

Spectral abscissa. The *spectral abscissa* of the system is denoted by $\alpha(\Delta)$ and defined as

$$\alpha(\Delta) := \max\{\Re(s) : \Delta(s) = 0, s \in \mathbb{C}\}.$$

Remark 2.9. The spectral abscissa determines the rightmost location of the characteristic roots. Therefore, it quantifies the dominant exponential decay rate of the closed-loop response. $\lrcorner$

DEFINITION
2.20

σ -stability. For a given $\sigma > 0$, the quasi-polynomial $\Delta(s; \mathbf{k}, h)$ is said to be σ -stable if

$$\alpha(\Delta) \leq -\sigma.$$

A straightforward approach for studying σ -stability consists of shifting the imaginary axis by σ units [38]. This is achieved through the change of variable

$$\hat{s} = s + \sigma, \quad (2.31)$$

where σ is a positive constant. After performing this transformation, the analysis reduces to studying the stability of the resulting quasi-polynomial, namely $P(\hat{s} - \sigma)$. Figure 2.4 illustrates this concept.

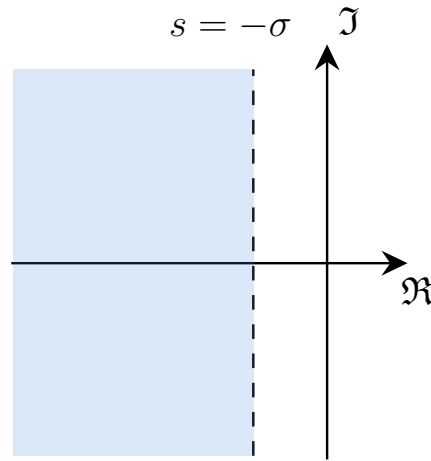
DEFINITION
2.21

Exponential stability. A retarded-type system is *exponentially stable* if there exist constants $M > 0$, $\sigma > 0$, such that

$$\|x(t)\| \leq M e^{-\sigma t} \|\phi\|, \quad t \geq 0,$$

for every admissible initial function ϕ .

Remark 2.10. For retarded-type systems, exponential stability is equivalent to all characteristic roots lying in the open LHP. $\lrcorner$

Figure 2.4 / Region of solutions of $P(s)$.

DEFINITION
2.22

Dominant roots. Let $\Delta(s) = 0$ be the characteristic equation of a system. The roots satisfying

$$\Re(s_i) = \alpha(\Delta)$$

are called dominant characteristic roots.

Remark 2.11. Dominant roots determine the asymptotic behavior of the system response. ┘

COROLLARY
2.1

If at least one pole satisfies

$$\Re(s_i) > 0,$$

then the system is unstable.

2.4 Continuity: The fundamental property

For systems involving time delays, the characteristic equation generally takes the form of a quasi-polynomial. Unlike ordinary polynomials, such expressions typically admit infinitely many roots. Consequently, stability is determined by examining how these roots are distributed over the complex plane.

An important aspect of this analysis is that the characteristic roots vary continuously with the system parameters. Hence, sufficiently small parameter perturbations lead to correspondingly small displacements of the roots. The results presented next provide a formal statement of this property.

First, it is convenient to introduce the following result, which will later be used to establish the continuity of the roots of quasi-polynomials:

THEOREM
2.2

Rouché's Theorem [49]. Let $f(z)$ and $g(z)$ be analytic functions inside and on a simple closed contour $\mathcal{C}^a$. If

$$|g(z)| < |f(z)|, \quad \forall z \in \mathcal{C}, \quad (2.32)$$

then the functions $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside $\mathcal{C}$.

^aIn complex analysis, a simple contour is a differentiable contour that does not present self-intersections.

Figure 2.5 illustrates the geometric interpretation of Rouché's theorem. The roots

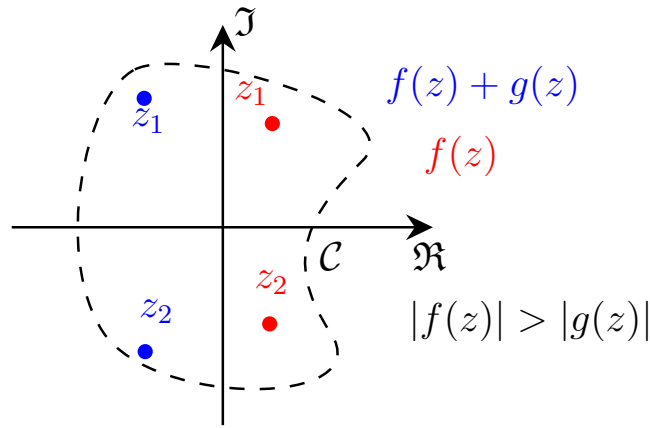


Figure 2.5 / Contour $\mathcal{C}$ for which the inequality $|f(z)| > |g(z)|$ holds.

of $f(z)$ located inside the contour are shown in red, while the roots of $f(z) + g(z)$ are shown in blue, which also remain inside the contour [50].

With Rouché's theorem in mind, consider the following result [49] [2] [3]

THEOREM
2.3

Let F, D be polynomials and let P be the quasi-polynomial defined by

$$\begin{aligned} P(s) &:= F(s) + D(s)e^{-hs}, & h &\geq 0, \\ F(s) &:= f_0 + f_1s + \cdots + f_ns^n, & f_n &\neq 0, \\ D(s) &:= d_0 + d_1s + \cdots + d_ms^m, \end{aligned}$$

where $n > m$. Define also

$$\begin{aligned} Q(s) &:= F_q(s) + D_q(s)e^{-hs}, \\ F_q(s) &:= (f_0 + \varepsilon_0) + (f_1 + \varepsilon_1)s + \cdots + (f_n + \varepsilon_n)s^n, \\ D_q(s) &:= (d_0 + \delta_0) + (d_1 + \delta_1)s + \cdots + (d_m + \delta_m)s^m, \end{aligned}$$

where $\varepsilon_j, \delta_\ell \in \mathbb{R}$ for all $j \in \{0, 1, \dots, n\}$ and $\ell \in \{0, 1, \dots, m\}$, with $f_n + \varepsilon_n \neq 0$.

Consider a circle $\mathcal{C}_k$ with radius $r_k \in \mathbb{R}_+$ and center $s_k \in \mathbb{C}$, where s_k is a root of $P(s)$ with multiplicity τ_k . Assume additionally that r_k is fixed and satisfies

$$0 < r_k < \min |s_k - s_j|, \quad j = 1, 2, \dots, k-1, k+1, \dots, m. \quad (2.33)$$

Then, there exist positive constants ε and δ , with

$$\varepsilon > |\varepsilon_i|, \quad i \in \{1, 2, \dots, n\},$$

and

$$\delta > |\delta_j|, \quad j \in \{1, 2, \dots, m\},$$

such that $Q(s)$ has exactly τ_k zeros inside $\mathcal{C}_k$.

Proof. Let $P(s)$ be defined as in Theorem 2.3, and define

$$\begin{aligned} R(s) &= F_r(s) + D_r(s)e^{-hs}, \\ F_r(s) &= \varepsilon_0 + \varepsilon_1 s + \dots + \varepsilon_n s^n, \\ D_r(s) &= \delta_0 + \delta_1 s + \dots + \delta_m s^m. \end{aligned}$$

Observe that

$$Q(s) = P(s) + R(s), \quad (2.34)$$

which can be interpreted as perturbed polynomials, or equivalently, as polynomials affected by noise.

Since $P(s)$ is continuous and nonzero for every $s \in \mathcal{C}_k$, there exists a positive constant μ_k such that

$$0 < \mu_k \leq |P(s)|, \quad \forall s \in \mathcal{C}_k.$$

On the other hand,

$$\begin{aligned} |R(s)| &= |F_r(s) + D_r(s)e^{-hs}| \\ &\leq |F_r(s)| + |D_r(s)e^{-hs}| \\ &= \sum_{i=0}^n |\varepsilon_i| |s|^i + e^{-h\Re(s)} \sum_{j=0}^m |\delta_j| |s|^j. \end{aligned}$$

Using the bounds

$$|\varepsilon_i| < \varepsilon, \quad |\delta_j| < \delta,$$

it follows that

$$|R(s)| \leq \varepsilon \sum_{i=0}^n |s|^i + \delta e^{-h\Re(s)} \sum_{j=0}^m |s|^j.$$

↳ Since $\mathcal{C}_k$ is compact in the complex plane, there exists a constant $\gamma > 0$ satisfying

$$e^{-h\Re(s)} \leq \gamma, \quad \forall s \in \mathcal{C}_k.$$

Moreover, for every $s \in \mathcal{C}_k$,

$$|s| = |(s - s_k) + s_k| \leq |s - s_k| + |s_k| = r_k + |s_k|.$$

Therefore,

$$|R(s)| \leq \varepsilon \sum_{i=0}^n (r_k + |s_k|)^i + \delta \gamma \sum_{j=0}^m (r_k + |s_k|)^j.$$

Now define

$$\varepsilon_m = \max\{\varepsilon, \delta\}.$$

Then,

$$|R(s)| \leq \varepsilon_m \left(\sum_{i=0}^n (r_k + |s_k|)^i + \gamma \sum_{j=0}^m (r_k + |s_k|)^j \right).$$

Hence, choosing ε_m such that

$$\varepsilon_m < \frac{\mu_k}{\sum_{i=0}^n (r_k + |s_k|)^i + \gamma \sum_{j=0}^m (r_k + |s_k|)^j},$$

guarantees that

$$|R(s)| < |P(s)|, \quad \forall s \in \mathcal{C}_k.$$

Finally, by a direct application of Rouché's theorem, the functions $Q(s) = P(s) + R(s)$ and $P(s)$ have the same number of zeros inside $\mathcal{C}_k$. Therefore, $Q(s)$ possesses exactly τ_k zeros inside $\mathcal{C}_k$. $\square$

As a direct consequence of the previous result, the following corollary is obtained:

COROLLARY

2.2

Let

$$P(s) = F(s) + D(s)e^{-hs} \quad (2.35)$$

be a quasi-polynomial with roots $s_1, s_2, \dots, s_k$ of multiplicities $\tau_1, \tau_2, \dots, \tau_k$, respectively. Then, there exist circles $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ such that, under sufficiently small perturbations in the parameters of $P(s)$, the number of roots inside each circle $\mathcal{C}_\ell$ remains exactly τ_ℓ , for every $\ell \in \{1, 2, \dots, k\}$.

Theorem 2.3 and Corollary 2.2 are illustrated in Figure 2.6. The red curves

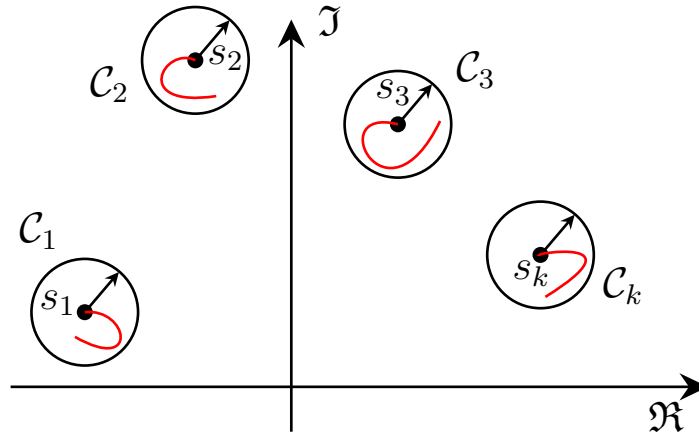


Figure 2.6 / Some characteristic roots associated with the quasi-polynomial $P(s)$ are represented in the complex plane.

illustrate the movement of these roots within the circles $\mathcal{C}_\ell$, with $\ell \in \{1, 2, 3, \dots, k\}$, as the perturbation in the parameters of $P(s)$ increases.

DEFINITION

2.23

Multiplicity of a characteristic root. A characteristic root s_0 of $\Delta(s)$ has *multiplicity* m if

$$\Delta(s_0) = \Delta'(s_0) = \dots = \Delta^{(m-1)}(s_0) = 0,$$

and

$$\Delta^{(m)}(s_0) \neq 0.$$

If $m = 1$, the root is called simple.

THEOREM

2.4

Let $\mathbb{C}$ denote the complex plane, and let $\mathcal{S}$ be an arbitrary open set. Then, the set $\mathcal{S}$, its boundary $\partial\mathcal{S}$, and the interior $\mathcal{U}^0$ of the closed set

$$\mathcal{U} = \mathbb{C} \setminus \mathcal{S}, \quad (2.36)$$

form a partition of the complex plane. That is,

$$\mathcal{S} \cup \partial\mathcal{S} \cup \mathcal{U}^0 = \mathbb{C}, \quad (2.37)$$

and

$$\mathcal{S} \cap \mathcal{U}^0 = \mathcal{S} \cap \partial\mathcal{S} = \partial\mathcal{S} \cap \mathcal{U}^0 = \emptyset. \quad (2.38)$$

Assume throughout that every set under consideration is nonempty. Let $P(\lambda, s)$ denote a collection of degree- n polynomials whose coefficients vary continuously with the parameter λ , where λ is restricted to a prescribed interval.

THEOREM

2.5

Boundary Crossing Theorem [49]. Suppose that $P(\lambda, s)$ is a family of polynomials of fixed degree n , continuously dependent on the parameter λ . Assume that $P(\lambda_1, s)$ has all its roots in $\mathcal{S}$, while $P(\lambda_2, s)$ has at least one root in $\mathcal{U}$. Then, there exists at least one value $\rho \in (\lambda_1, \lambda_2)$ such that one of the following conditions holds:

- (1) $P(\rho, s)$ has roots in $\mathcal{S} \cap \partial\mathcal{S}$,
- (2) $P(\rho, s)$ has at least one root on $\partial\mathcal{S}$.

The proofs of the previous two theorems rely on Rouché's theorem and its corresponding corollary. For additional details, the reader is referred to [49]. Based on these results, the continuity property of the roots can now be formally introduced through the following theorem.

THEOREM

2.6

Continuity of paramaters. Let $\Delta(s, \lambda)$ be a retarded quasi-polynomial continuously depending on the parameter vector $k_1 \in \mathbb{R}^m$. Then, the characteristic roots of

$$\Delta(s, \lambda) = 0$$

depend continuously on λ .

Example 2.1

Consider the first-order polynomial

$$p(s) = \lambda_1 s + \lambda_2, \quad (2.39)$$

whose parameters belong to the admissible region

$$\mathcal{P} := [\lambda_1, \lambda_2]. \quad (2.40)$$

Suppose that there exists at least one parameter combination in $\mathcal{P}$ for which $p(s)$ is Hurwitz.

Next, let the parameter vector move from the initial point $(\lambda_{1o}, \lambda_{2o})$ to the final point $(\lambda_{1f}, \lambda_{2f})$ following the trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$ illustrated in Figures 2.9–2.8.

Along the trajectory $\mathcal{T}_1$, the continuity property is preserved, meaning that no degree loss occurs despite the parametric variation. Consequently, the root trajectories

$$s_i(\lambda_1, \lambda_2), \quad i \in \{1, \dots, n\},$$

also evolve continuously. In particular, the polynomial has the root $s = 0$ when $\lambda_1 = 0$, and therefore a boundary crossing takes place.

In contrast, if the parameters move along the trajectory $\mathcal{T}_2$, discontinuities arise because a root reaches $i\omega$ for a finite value of ω while the polynomial simultaneously loses degree. Consequently, no boundary crossing takes place.

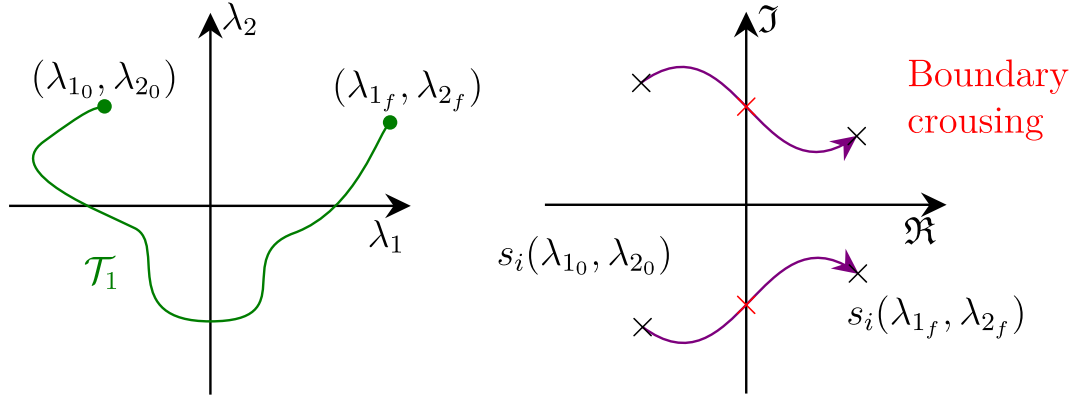


Figure 2.7 / Parameter-space representation and root-locus behavior along $\mathcal{T}_1$.

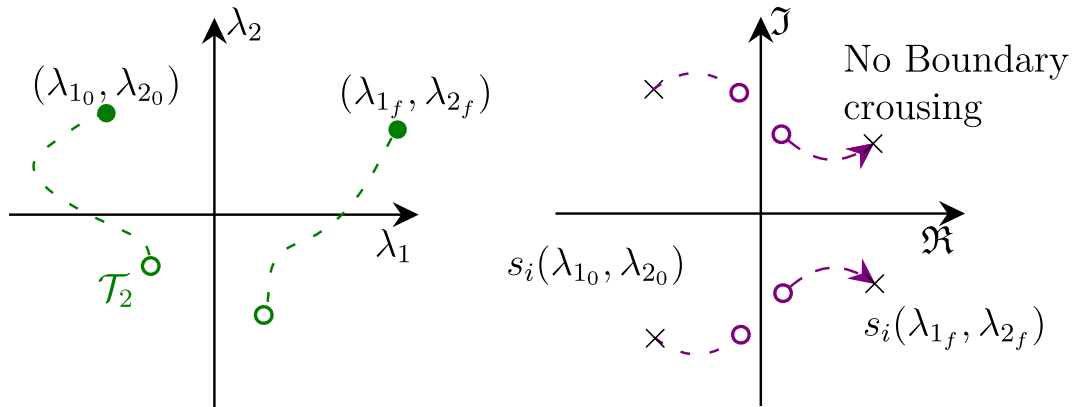


Figure 2.8 / Parameter-space representation and root-locus behavior along $\mathcal{T}_2$.

The results developed above establish a general mathematical basis for identifying stability boundaries without relying exclusively on conventional polynomial stability criteria or standard tests [2] [3] [24]. Since the formulation is entirely mathematical, its application must be tailored to the specific characteristics of the system being analyzed. For linear systems, the resulting conditions can often be interpreted geometrically and represented through algebraic curves; see, for instance, [51]. This connection also reflects certain aspects associated with *Hilbert's 16th Problem*.

DEFINITION

2.24

Crossing frequency. A *crossing frequency* is a value $\omega \in \mathbb{R}_+$ such that the characteristic equation has roots on the imaginary axis, that is,

$$\Delta(i\omega) = 0.$$

DEFINITION
2.25

Stability crossing curve. A *stability crossing curve* consists of the controller parameter values that place one or more characteristic roots on the imaginary axis.

Two stability boundaries are especially relevant:

- The *Complex Root Boundary* (CRB), associated with nonzero imaginary roots

$$s = \pm i\omega, \quad \omega > 0.$$

- The *Real Root Boundary* (RRB), associated with crossings through the origin

$$s = 0.$$

The preceding developments make it possible to establish boundary-crossing behavior for parameter trajectories that preserve the required continuity conditions [2] [3] [50] [52]. However, an interesting question arises when considering trajectories that begin and end at the same point, as illustrated in Figure 2.9.

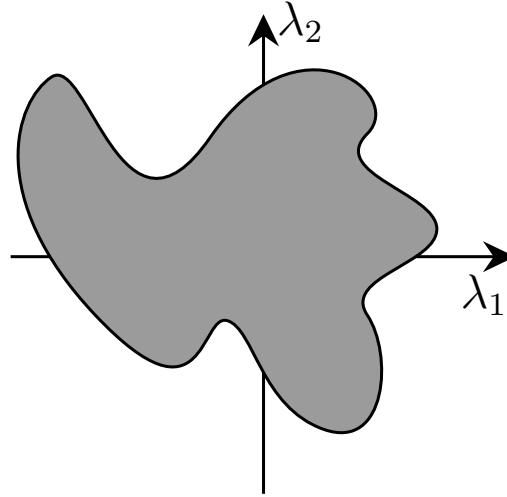


Figure 2.9 / Closed path.

In order to characterize the interior of the parametric plane generated by a closed path, it is useful to revisit several foundational ideas from the theory of $\mathcal{D}$ -decomposition. The works presented in [37] and [53] provide important motivations, while the first systematic stability-domain analysis for a third-order polynomial with two uncertain parameters was introduced in [53]. Some relevant historical developments are summarized below:

- *Neimark* popularized the $\mathcal{D}$ -decomposition approach by developing a method for stability-domain analysis in 1947 [54] [55].
- The books *Robust Control: The Parameter Space Approach* [3] and *Nonlinear Systems: Analysis and Design* [56], by Ackermann and Siljak, respectively, were among the first Western references addressing this methodology.

- As stated in *Analytical Design of PID Controllers* [57] by Díaz-Rodríguez *et al.*, “its use for the design of low-order controllers for linear systems is very efficient.”

Combined with the boundary crossing theorem, these developments provide a basis for characterizing stability regions and constitute a relevant contribution to robust control from a parameter-space perspective.

To describe the $\mathcal{D}$ -decomposition methodology, we follow the definition presented in [58], where the Index Property is employed as a graphical tool for characterizing stability domains.

DEFINITION
2.26

$\mathcal{D}$ -decomposition [37]. Consider a linear system whose characteristic equation is given by

$$\sum_{i=1}^n \lambda_i G_i(s) = 0, \quad (2.41)$$

where $s = \sigma + j\omega$ is the Laplace complex variable, the functions $G_i(s)$ are rational functions, and $\lambda_i \in \mathbb{R}$ are real-valued parameters.

The n -dimensional parameter space

$$\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]^\top$$

can be partitioned into subsets denoted by

$$D(k, N - k),$$

which correspond to the regions where the characteristic equation (2.41) possesses k roots in the left half-plane (LHP) and $N - k$ roots in the right half-plane (RHP), where N is the total number of roots of the characteristic equation.

The system is stable for those parameter values that belong to the region

$$D(N, 0).$$

Such a partition of the parameter space Λ into the sets $D(k, N - k)$ is called the $\mathcal{D}$ -decomposition.

Consider a polynomial of the form $P(s, \mathcal{P})$, of degree n , where $n \in \mathbb{Z}_+$ and $\mathcal{P} \in \mathbb{R}^m$ denotes the set of polynomial parameters $(p_1, p_2, \dots, p_m)$.

For a fixed parameter vector $\mathcal{P}^*$, suppose that the polynomial possesses k unstable roots and $n - k$ stable roots. Then, the domain to which $\mathcal{P}^*$ belongs is denoted by $\mathcal{D}(k)$, such that $\mathcal{D}(0)$ corresponds to the stability domain [37].

In this way, the method allows the parameter space to be decomposed into different domains characterized by the number of stable roots contained in each region.

Consider the family of polynomials with two parameters

$$P(s, p_1, p_2) = \{Q(s) + p_1 D(s) + p_2 F(s), \quad p_1, p_2 \in \mathbb{R}\}, \quad (2.42)$$

where $Q(s)$, $D(s)$, and $F(s)$ are polynomials of degree n .

The $\mathcal{D}$ -partition curve is then defined by the equation

$$P(j\omega, p_1, p_2) = 0.$$

Solving for the parameters yields

$$p_1 = -\frac{\Delta_1}{\Delta}, \quad p_2 = -\frac{\Delta_2}{\Delta}, \quad (2.43)$$

where

$$\Delta = \begin{vmatrix} U_D & U_F \\ V_D & V_F \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} U_Q & U_F \\ V_Q & V_F \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} U_D & U_Q \\ V_D & V_Q \end{vmatrix}. \quad (2.44)$$

The functions above are defined as

$$\begin{aligned} U_Q &= q_0 - q_2\omega^2 + q_4\omega^4 + \cdots, \\ U_D &= d_0 - d_2\omega^2 + d_4\omega^4 + \cdots, \\ U_F &= f_0 - f_2\omega^2 + f_4\omega^4 + \cdots, \\ V_Q &= q_1 - q_3\omega^2 + q_5\omega^4 + \cdots, \\ V_D &= d_1 - d_3\omega^2 + d_5\omega^4 + \cdots, \\ V_F &= f_1 - f_3\omega^2 + f_5\omega^4 + \cdots. \end{aligned}$$

For $\Delta \neq 0$, equations (2.43) describe a parametric curve. When this curve is crossed, two roots move from one side of the complex plane to the other. Furthermore, the $\mathcal{D}$ -partition also contains the lines

$$q_0 + p_1 d_0 + p_2 f_0 = 0, \quad q_n + p_1 d_n + p_2 f_n = 0. \quad (2.45)$$

Notice that these correspond to the particular case $\omega = 0$, which implies that the curve described by (2.43) starts on one of these lines and ends on another. Crossing these lines causes a single root to move from one half-plane to the other.

If $\Delta = 0$ and either $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$, the curve described by (2.43) extends to infinity, thus generating an asymptotic curve. The value of ω for which

$$\Delta = \Delta_1 = \Delta_2 = 0$$

is called an *isolated frequency*. In this particular case, the $\mathcal{D}$ -partition does not contain a point but rather a line, which is referred to as a *singular line*.

Example 2.2: $\mathcal{D}$ -partition with two parameters

Consider the polynomial

$$P(s, p_1, p_2) = 4s^2 + s + p_1 s e^{-s} + p_2 e^{-s}, \quad (2.46)$$

with

$$Q(s) = 4s^2 + s,$$

$$D(s) = s e^{-s},$$

$$F(s) = e^{-s}.$$

Using the $\mathcal{D}$ -partition method, the partition of the parameter plane (p_1, p_2) is obtained, as illustrated in Figure 2.10. The blue curves correspond both to the boundary curve

$$P(j\omega, p_1, p_2) = 0$$

and the red curve the singular line.

Notice that, according to the previous discussion, crossing the curve $P(j\omega, p_1, p_2) = 0$ causes two roots to move from one side of the complex plane to the other, whereas crossing a singular line causes only one root to cross. As expected, crossing both the curve and the singular line simultaneously involves three distinct roots.

Two important features can be highlighted from Figure 2.10:

- (1) The region $\mathcal{D}(0)$, also referred to as the stability domain, where no unstable roots exist.
- (2) The directions in which the curve and the singular lines are crossed directly affect the manner in which roots are gained or lost. This property will be exploited in the following chapter in order to characterize the behavior of the roots while crossing the boundary curve that delimits the stable region.

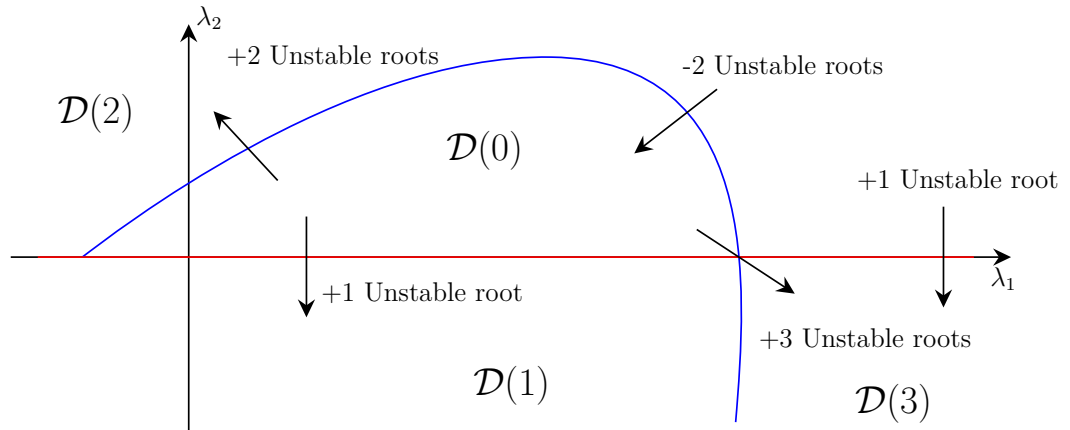


Figure 2.10 / D-Partitions in the (p_1, p_2) parameter plane.

Remark 2.12. Since roots of retarded quasi-polynomials move continuously with respect to parameters, the number of unstable roots can change only when roots cross the imaginary axis. This is the principle behind CRB, RRB, and $\mathcal{D}$ -partition methods. $\lrcorner$

Finally, the substitution

$$s = -\sigma + i\omega$$

is used to characterize σ -stability boundaries. In this case, the condition

$$\alpha(\Delta) \leq -\sigma$$

means that all characteristic roots lie to the left of the vertical line $\Re(s) = -\sigma$.

DEFINITION

2.27

Fragility. A controller is said to be *fragile* if small perturbations in its parameters or delays may destabilize the closed-loop system.

Remark 2.13. *Fragility* is closely related to the geometry of the stability regions and the proximity of the characteristic roots to the stability boundary. $\lrcorner$

2.5 Fundamental Mathematics for Robotics

All mathematical preliminaries and theoretical foundations presented in this work are based on the references [59], [60], [61], and [62].

The following sections provide a review of the position, velocity, and acceleration analyses for open kinematic chains, such as serial robots. Furthermore, the notation adopted throughout this work follows the standard conventions commonly used in robotics courses and literature.

Rigid-Body Kinematics: Transformations

Let $\mathcal{B}$ denote a rigid body whose motion takes place in the three-dimensional Euclidean space, as illustrated in Figure 2.11. Define the spatial reference frame as

$$\Sigma_{\text{spatial}} = (O_{\text{spatial}}, \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}), \quad (2.47)$$

which remains fixed with respect to the surrounding space. Likewise, introduce the body-fixed frame

$$\Sigma_{\text{body}} = (O_{\text{body}}, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}), \quad (2.48)$$

whose origin and basis move together with the rigid body. Initially, the bases

$$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\} \quad \text{and} \quad \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \quad (2.49)$$

are assumed to be aligned with one another.

Let $\mathbf{r} \in \mathbb{R}^3$ denote the vector connecting the origin O_{spatial} of the frame Σ_{spatial} to the origin O_{body} of the frame Σ_{body} , i.e.,

$$\mathbf{r} = O_{\text{body}} - O_{\text{spatial}}.$$

Furthermore, let $\mathbf{R} \in \mathbb{R}^{3 \times 3}$ be the rotation matrix whose a -th column contains the components of the vector $\mathbf{b}_a$ with respect to the basis

$$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}.$$

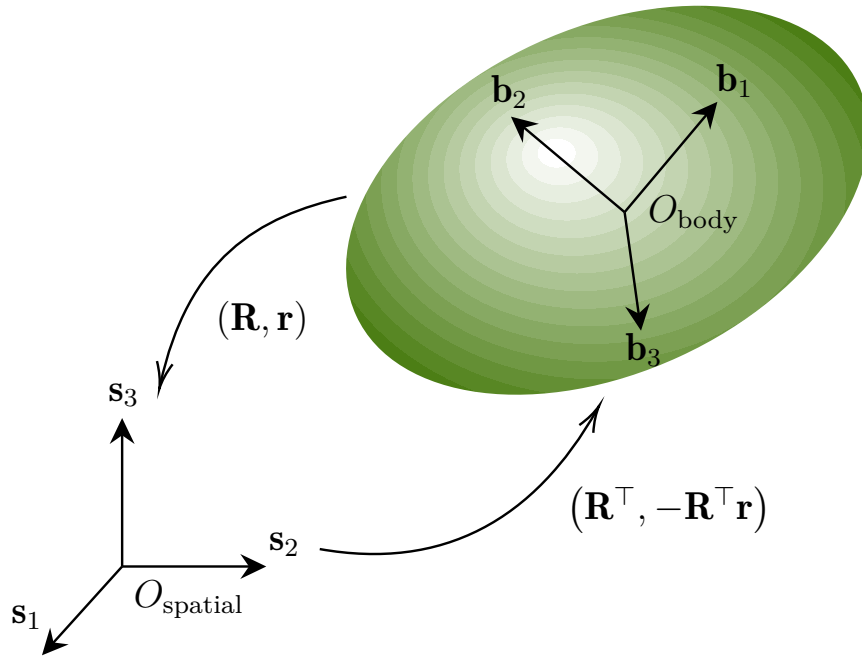


Figure 2.11 / Reference frame transformations.

The position of a point in space may be expressed with respect to either the spatial frame or the body-fixed frame. Let the corresponding coordinate representations be denoted by χ_s and χ_b , respectively. These two vectors are connected by an affine coordinate transformation

$$\chi_s = \mathbf{R}\chi_b + \mathbf{r}. \quad (2.50)$$

It is often convenient to express this transformation in an equivalent homogeneous form.

DEFINITION

2.28

Homogeneous coordinates [6]. For $\mathbf{x} \in \mathbb{R}^3$, define

$$\bar{\mathbf{x}} = (\mathbf{x}, 1) \in \mathbb{R}^4$$

as the *homogeneous coordinates* of $\mathbf{x}$.

Using this notation, transformation (2.50) and its inverse can be rewritten as

$$\bar{\chi}_s = \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \bar{\chi}_b, \quad \bar{\chi}_b = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T \mathbf{r} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \bar{\chi}_s, \quad (2.51)$$

where $\mathbf{0}_{m \times n}$ denotes the zero matrix in $\mathbb{R}^{m \times n}$. Both transformations are illustrated in Figure 2.11.

DEFINITION
2.29

Orthogonal group [6]. The *orthogonal group* $O(n)$ is defined as

$$O(n) = \left\{ \mathbf{R} \in \mathbb{R}^{n \times n} \mid \mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}_n \right\}. \quad (2.52)$$

DEFINITION
2.30

Special orthogonal group [6]. The *special orthogonal group* $SO(n)$ is defined as

$$SO(n) = \{ \mathbf{R} \in O(n) \mid \det(\mathbf{R}) = 1 \}. \quad (2.53)$$

Remark 2.14. For $n = 3$, $SO(3)$ is the set used to represent rotations in $\mathbb{R}^3$. A matrix $\mathbf{R} \in SO(3)$ that gives the orientation of the body-fixed frame relative to the spatial frame is referred to as a *rotation matrix*. $\lrcorner$

The pose of a rigid body can be represented using homogeneous coordinates through the matrix

$$\mathbf{g} = \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}_{1 \times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}.$$

Motivated by this representation,

DEFINITION
2.31

Euclidean group [6]. The *Euclidean group* is defined as

$$E(n) = \left\{ \mathbf{g} \in \mathbb{R}^{(n+1) \times (n+1)} \mid \mathbf{g} = \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}_{1 \times n} & 1 \end{bmatrix}, \mathbf{R} \in O(n), \mathbf{r} \in \mathbb{R}^n \right\}. \quad (2.54)$$

Under matrix multiplication, the collection of matrices $\mathbf{g}$ defines a Lie group called the *special Euclidean group*, denoted by $SE(n)$.

DEFINITION
2.32

Special Euclidean Group [6]. The *special Euclidean group* and is denoted by $SE(n)$. It is defined as

$$SE(n) = \{ \mathbf{g} \in E(n) \mid \det(\mathbf{g}) = 1 \}. \quad (2.55)$$

Remark 2.15. For $n = 3$, the set $\text{SE}(3)$ corresponds to the *group of rigid-body displacements* in $\mathbb{R}^3$. $\lrcorner$

A matrix $\mathbf{g} \in \text{SE}(3)$ describing both the orientation and position of the body frame with respect to the spatial frame is referred to as a *rigid-body transformation matrix* or *homogeneous transformation matrix*.

It is convenient to identify the elements of $\text{SO}(n) \times \mathbb{R}^n$ with those of $\text{SE}(n)$ through the bijection³

$$(\mathbf{R}, \mathbf{r}) \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ \mathbf{0}_{1 \times n} & 1 \end{bmatrix}.$$

Remark 2.16. The matrix sets $\text{O}(n)$, $\text{SO}(n)$, $\text{E}(n)$, and $\text{SE}(n)$ remain closed^a with respect to both matrix multiplication and matrix inversion. $\lrcorner$

^aA set is *closed* when it is equal to its closure, that is, $\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}'$.

Examining the elements of $\text{O}(n)$ and $\text{E}(n)$ as linear and affine transformations on $\mathbb{R}^n$, respectively, reveals important geometric properties. In particular, orthogonal matrices, i.e., elements of $\text{O}(n)$, preserve vector norms and therefore represent rotations and reflections. Similarly, the elements of $\text{E}(n)$ describe rigid transformations composed of rotations, translations, and reflections.

In many robotics problems, reflections are not considered, so the elements of $\text{O}(n)$ are restricted to those with positive determinant. If the orthogonal matrix describes the orientation of the body-fixed frame with respect to the spatial frame, then the condition

$$\det(\mathbf{R}) > 0$$

means that the bases

$$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\} \quad \text{and} \quad \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$$

have the same orientation.

More precisely, regarding the property of preserving distances between points, a mapping

$$\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is called an *isometry*⁴ if

$$\|\phi(\mathbf{y}) - \phi(\mathbf{z})\|_{\mathbb{R}^n} = \|\mathbf{y} - \mathbf{z}\|_{\mathbb{R}^n}, \quad \forall \mathbf{y}, \mathbf{z} \in \mathbb{R}^n.$$

In the following, the elements

$$\mathbf{g} = (\mathbf{R}, \mathbf{r}) \in \text{E}(n)$$

³A function is said to be **bijective** if it is both one-to-one and onto, i.e., every element of the codomain corresponds to exactly one element of the domain.

⁴An *isometry* is a mapping between metric spaces that preserves distances between points.

are identified with the corresponding affine transformations

$$\phi_{\mathbf{g}} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \mathbf{y} \mapsto \mathbf{R}\mathbf{y} + \mathbf{r}.$$

PROPOSITION
2.1

Isometries of $\mathbb{R}^n$ [6]. A mapping $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry if and only if there exists an element

$$\mathbf{g} \in E(n)$$

such that

$$\phi = \phi_{\mathbf{g}}.$$

Proof. See [6]. □

The Euclidean group $E(n)$ corresponds to the group of isometries of $\mathbb{R}^n$, that is, the set of all distance-preserving transformations in $\mathbb{R}^n$. In this context, rotations and translations are referred to as *rigid-body transformations*.

Rigid-Body Kinematics: Infinitesimal Transformations

This section focuses on the velocity of rigid-body motion. In particular, the differential equations that describe linear and angular velocities are presented.

DEFINITION
2.33

Vector space of skew-symmetric matrices [6]. For $n \in \mathbb{N}$, let $\mathfrak{so}(n)$ denote the *vector space of skew-symmetric matrices* in $\mathbb{R}^{n \times n}$:

$$\mathfrak{so}(n) = \left\{ \mathbf{S} \in \mathbb{R}^{n \times n} \mid \mathbf{S}^T = -\mathbf{S} \right\}. \quad (2.56)$$

Let $\times$ denote the vector cross product in $\mathbb{R}^3$, and define the linear mapping

$$\widehat{(\cdot)} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$$

by

$$\widehat{\boldsymbol{\omega}} \mathbf{y} = \boldsymbol{\omega} \times \mathbf{y}, \quad \forall \boldsymbol{\omega}, \mathbf{y} \in \mathbb{R}^3.$$

Since the mapping $\widehat{(\cdot)}$ is an isomorphism⁵ between vector spaces, one writes

$$\mathfrak{so}(3) \simeq \mathbb{R}^3.$$

Furthermore, for future reference, it is convenient to define the inverse mapping

$$(\cdot)^\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3.$$

⁵An *isomorphism* is a bijective morphism that preserves the algebraic structure between two mathematical spaces.

For

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^\top,$$

one has

$$\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \quad \left(\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \right)^\vee = \boldsymbol{\omega}. \quad (2.57)$$

These notions can be extended in the following manner. Define

$$\mathfrak{se}(n) = \left\{ \begin{bmatrix} \mathbf{S} & \mathbf{v} \\ \mathbf{0}_{1 \times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)} \mid \mathbf{S} \in \mathfrak{so}(n), \mathbf{v} \in \mathbb{R}^n \right\}. \quad (2.58)$$

Analogously to the case of $\mathfrak{so}(3)$, it is convenient to introduce, with a slight abuse of notation, the linear mapping

$$\widehat{(\cdot)} : \mathbb{R}^3 \oplus \mathbb{R}^3 \rightarrow \mathfrak{se}(3)$$

defined by

$$\widehat{\boldsymbol{\xi}} = \begin{bmatrix} \widehat{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix}, \quad (2.59)$$

for

$$\boldsymbol{\xi} = (\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}^3 \oplus \mathbb{R}^3.$$

Since the mapping

$$\widehat{(\cdot)} : \mathbb{R}^3 \oplus \mathbb{R}^3 \rightarrow \mathfrak{se}(3)$$

is an isomorphism of vector spaces, one writes

$$\mathfrak{se}(3) \simeq \mathbb{R}^3 \oplus \mathbb{R}^3,$$

and defines

$$(\cdot)^\vee : \mathfrak{se}(3) \rightarrow \mathbb{R}^3 \oplus \mathbb{R}^3$$

as the corresponding inverse isomorphism.

DEFINITION
2.34

Twists [6]. The elements of $\mathfrak{se}(3)$ are commonly referred to as *twists*^a.

^aThe term is used in reference to screw theory; they are also known as 6D vectors.

DEFINITION
2.35

Tangent space [6]. The *tangent space*^a at the identity element of the *special Euclidean group* $\text{SE}(n)$ is defined as the Lie algebra $\mathfrak{se}(n)$ (see (2.58)).

^aThe *tangent space* associated with a point on a differentiable manifold is the vector space formed by all tangent vectors at that point. Its dimension coincides with the dimension of the manifold.

Remark 2.17. The spatial and body velocities describe the same physical motion, but they are expressed with respect to different reference frames. ┘

DEFINITION
2.36

Adjoint map [6]. Given a rigid body with configuration

$$\mathbf{g} \in \text{SE}(3),$$

the transformation from body-frame twists to spatial-frame twists is known as the *adjoint map*

$$\text{Ad}_{\mathbf{g}} : \mathfrak{se}(3) \rightarrow \mathfrak{se}(3),$$

and is defined by

$$\text{Ad}_{\mathbf{g}} \hat{\boldsymbol{\eta}} = \mathbf{g} \hat{\boldsymbol{\eta}} \mathbf{g}^{-1}. \quad (2.60)$$

Remark 2.18. Using the identification

$$\mathfrak{se}(3) \simeq \mathbb{R}^3 \oplus \mathbb{R}^3,$$

it can be shown that this definition is well posed and that the mapping

$$\text{Ad}_{\mathbf{g}} : \mathbb{R}^3 \oplus \mathbb{R}^3 \rightarrow \mathbb{R}^3 \oplus \mathbb{R}^3$$

can be represented as

$$\text{Ad}_{(\mathbf{R}, \mathbf{r})} = \begin{bmatrix} \mathbf{R} & \mathbf{0}_{3 \times 3} \\ \hat{\mathbf{r}}\mathbf{R} & \mathbf{R} \end{bmatrix}. \quad (2.61)$$

Lie Groups and Lie Algebras

DEFINITION
2.37

Group [6]. A set $\mathcal{G}$ equipped with a binary operation

$$\mathcal{G} \times \mathcal{G} \ni (a, b) \mapsto a \star b \in \mathcal{G},$$

is called a *group* if the following properties hold:

- (i) $a \star (b \star c) = (a \star b) \star c, \quad \forall a, b, c \in \mathcal{G},$ (*associativity*);
- (ii) There exists an element $e \in \mathcal{G}$ such that $a \star e = e \star a = a,$ $\forall a \in \mathcal{G},$ (*identity element*);

- (iii) For every $a \in \mathcal{G}$, there exists an element $a^{-1} \in \mathcal{G}$ such that $a \star a^{-1} = a^{-1} \star a = e$, (*inverse element*).

Remark 2.19. When needed, the notation $(\mathcal{G}, \star)$ is used to make the group operation explicit. If the operation can be understood from the context, the symbol $\star$ is omitted. $\lrcorner$

DEFINITION
2.38

Topological group [6]. A *topological group* is a group that is also a topological space for which the group operation and the inverse operation are continuous.

DEFINITION
2.39

Lie group [6]. A *Lie group* is a topological group that is additionally a differentiable manifold, where both the group operation and the inverse operation are smooth mappings.

DEFINITION
2.40

Abelian group [6]. A group is said to be *abelian* (or commutative) if the group operation is commutative, that is,

$$a \star b = b \star a, \quad \forall a, b \in \mathcal{G}.$$

There exist important examples of both abelian and non-abelian Lie groups.

Example 2.3

Typical examples of abelian groups are $(\mathbb{Z}, +)$, $(\mathbb{Q} \setminus \{0\}, \cdot)$, and $(\mathbb{R}_{>0}, \cdot)$.

Example 2.4

The collection of invertible real matrices of dimension $n \times n$, denoted by $\text{GL}(n; \mathbb{R})$, constitutes a Lie group with respect to matrix multiplication. It is commonly referred to as the *general linear group over $\mathbb{R}$* , or, in abbreviated form, the *general linear group*.

Example 2.5

The neutral element of this group is the identity matrix $\mathbf{I}_n$. For any matrix $\mathbf{A} \in \text{GL}(n; \mathbb{R})$, the corresponding group inverse is the usual matrix inverse $\mathbf{A}^{-1}$.

Example 2.6

For $n > 1$, $\text{GL}(n; \mathbb{R})$ does not satisfy the commutative property, since the product of two matrices depends on their order in general. Hence, $\text{GL}(n; \mathbb{R})$ is a non-abelian group.

Let $(\mathcal{G}, \star)$ and $(\mathcal{H}, \bullet)$ be two groups.

DEFINITION
2.41

Group homomorphism [6]. A *group homomorphism* is a mapping

$$\rho : \mathcal{G} \rightarrow \mathcal{H}$$

satisfying

$$\rho(a \star b) = \rho(a) \bullet \rho(b), \quad \forall a, b \in \mathcal{G}.$$

DEFINITION
2.42

Lie group homomorphism [6]. A *Lie group homomorphism* is a smooth homomorphism between Lie groups.

A group homomorphism

$$\rho : \mathcal{G} \rightarrow \mathcal{H}$$

preserves both identity and inverse elements. Thus, the identity of $\mathcal{G}$ is mapped to the identity of $\mathcal{H}$, while the inverse of any element in $\mathcal{G}$ is mapped to the inverse of its image in $\mathcal{H}$.

Remark 2.20. If the homomorphism ρ is bijective, it is called an *isomorphism*. In that case, the groups $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic*, which is denoted by

$$\mathcal{G} \simeq \mathcal{H}.$$

┘

Example 2.7

The adjoint mapping

$$\mathbf{g} \mapsto \text{Ad}_{\mathbf{g}},$$

defined previously, is a Lie group homomorphism from $\text{SE}(3)$ into the set of invertible linear mappings on $\mathfrak{se}(3)$.

Remark 2.21. Group homomorphisms play a role similar to that of linear maps between vector spaces. In both cases, the corresponding algebraic structure is preserved: linear maps preserve the structure of vector spaces, while group homomorphisms preserve the structure of groups.

┘

DEFINITION

2.43

Subgroup. Let $\mathcal{G}$ be a group.

- (i) A *subgroup* $\mathcal{H}$ of $\mathcal{G}$ is a subset of $\mathcal{G}$ that forms a group under the group operation defined on $\mathcal{G}$.

Now, let $\mathcal{G}$ be a Lie group.

- (ii) A *Lie subgroup* of $\mathcal{G}$ is a subgroup $\mathcal{H} \subseteq \mathcal{G}$ such that the inclusion map $i_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{G}$ is an injective immersion;
- (iii) A Lie subgroup $\mathcal{H}$ of $\mathcal{G}$ that is also a *submanifold* of $\mathcal{G}$ is called a *regular Lie subgroup*.

Remark 2.22. In other words, a subgroup $\mathcal{H}$ of a group $\mathcal{G}$ is closed under the group operation, closed under the inverse operation, and contains the identity element. $\lrcorner$

With these notions, the following concept can also be introduced.

DEFINITION

2.44

Matrix Lie Group. A *matrix Lie group* is a subgroup of $\mathrm{GL}(n; \mathbb{R})$.

Note that

PROPOSITION

2.2

A matrix Lie group is an immersed submanifold of $\mathrm{GL}(n; \mathbb{R})$, and that it is an embedded submanifold of $\mathrm{GL}(n; \mathbb{R})$ if and only if it is topologically closed.

Remark 2.23. An equivalent way to define a subgroup is sometimes useful. The pair $(\mathcal{K}, \rho)$ is called a subgroup of $\mathcal{G}$ when $\mathcal{K}$ is itself a group and

$$\rho : \mathcal{K} \rightarrow \mathcal{G}$$

is an injective group homomorphism.

With this definition, a subgroup is considered up to group isomorphism. $\lrcorner$

Example 2.8

The rotation group $\mathrm{SO}(3)$ and the rigid-body displacement group $\mathrm{SE}(3)$ are examples of matrix Lie groups.

Example 2.9

The groups

$$\mathrm{SO}(3) \subset \mathrm{GL}(3; \mathbb{R}) \quad \text{and} \quad (\mathbb{R}^3, +)$$

are isomorphic to subgroups of the rigid-body displacement group

$$\text{SE}(3) \subset \text{GL}(4; \mathbb{R})$$

through the embeddings

$$\text{SO}(3) \ni \mathbf{R} \mapsto \begin{bmatrix} \mathbf{R} & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \in \text{SE}(3),$$

and

$$\mathbb{R}^3 \ni \mathbf{v} \mapsto \begin{bmatrix} \mathbf{I}_3 & \mathbf{v} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \in \text{SE}(3).$$

All the concepts presented above have been introduced from a mathematical perspective. Using the notation commonly adopted in robotics, the following formulations are obtained.

Robotics: Position

Position analysis in serial robotics is performed through the use of *homogeneous transformation matrices*,

$$\mathbf{T} \in \mathbb{R}^{4 \times 4} \quad (\mathbf{T} \in \text{SE}(3)).$$

These matrices describe the relative *position* and *orientation* between two different reference frames attached to the rigid bodies composing the kinematic chain. In addition, they incorporate the notion commonly referred to as *perspective* in robotics.

Using the standard subscript notation,

DEFINITION 2.45

Homogeneous transformation matrix. The *homogeneous transformation matrix* between two consecutive reference frames, namely frames $i - 1$ and i , is defined as

$$\mathbf{T}_{i-1,i} = \begin{bmatrix} \mathbf{R}_{i-1,i} & \mathbf{r}_{i-1,i} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \quad (2.62)$$

where $\mathbf{R}_{i-1,i} \in \mathbb{R}^{3 \times 3}$ ($\mathbf{R}_{i-1,i} \in \text{SO}(3)$) denotes the rotation matrix between consecutive reference frames, and

$$\mathbf{r}_{i-1,i} \in \mathbb{R}^3$$

represents the position vector of the center of mass expressed in the body reference frame.

As discussed previously, the definition of a *homogeneous transformation matrix* requires both a rotation matrix and a translation vector.

DEFINITION
2.46

Inverse homogeneous transformation matrix. The *inverse homogeneous transformation matrix* is given by

$$\mathbf{T}_{i-1,i}^{-1} = \begin{bmatrix} \mathbf{R}_{i-1,i}^\top & -\mathbf{R}_{i-1,i}^\top \mathbf{r}_{i-1,i} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (2.63)$$

In this case, the following relation between the position vectors connecting the two reference frames has been introduced:

$$\mathbf{r}_{i,i-1} = -\mathbf{R}_{i-1,i}^\top \mathbf{r}_{i-1,i} = -\mathbf{R}_{i,i-1} \mathbf{r}_{i-1,i}.$$

DEFINITION
2.47

Rotation matrix. The *rotation matrices* are defined according to the axis about which the reference frame rotates, namely:

- Rotation about the x -axis: $\mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$
- Rotation about the y -axis: $\mathbf{R}_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$
- Rotation about the z -axis: $\mathbf{R}_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The position analysis of an open kinematic chain is obtained by consecutively multiplying all homogeneous transformation matrices from the robot base frame to the end-effector frame. The resulting transformation relating the two reference frames located at the ends of the kinematic chain can be expressed as

$$\mathbf{T}_{0,n} = \prod_{i=1}^n \mathbf{T}_{i-1,i}. \quad (2.64)$$

Between each pair of consecutive bodies, a joint is located, which may be either passive or actuated. Each joint possesses an associated homogeneous transformation matrix,

$$\mathbf{T}_{i-1,i},$$

which is generally not constant, since it depends on the joint degrees of freedom.

Consequently, two types of homogeneous transformation matrices appear in an open kinematic chain: those associated with the joints, which depend on the generalized coordinates, and constant transformation matrices related to the geometry and relative placement of the rigid bodies.

DEFINITION

2.48

The homogeneous transformation matrices associated with the most common joint types are defined as follows:

- *Revolute joint*. It is defined as a rotation about the z -axis:

$$\begin{bmatrix} \mathbf{R}_z(\theta) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}.$$

- *Prismatic joint*. It is defined as a translation parallel to the z -axis:

$$\begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_z(d) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}.$$

- *Cylindrical joint*. It is defined as a combined rotation and translation along the z -axis:

$$\begin{bmatrix} \mathbf{R}_z(\theta) & \mathbf{r}_z(d) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}.$$

Robotics: Velocity

For an open kinematic chain, the velocity analysis is obtained by differentiating the position equation (2.64). The resulting expression is commonly multiplied by the inverse homogeneous transformation matrix,

$$\mathbf{T}_{0,n}^{-1},$$

in order to express the velocities with respect to the reference frame attached to each body. The resulting expression is given by

$$\mathbf{T}_{0,n}^{-1} \dot{\mathbf{T}}_{0,n} = \sum_{i=1}^n \mathbf{T}_{i,n}^{-1} (\mathbf{T}_{i-1,i}^{-1} \dot{\mathbf{T}}_{i-1,i}) \mathbf{T}_{i,n}. \quad (2.65)$$

The product

$$\mathbf{T}^{-1} \dot{\mathbf{T}}$$

only appears in bodies connected through a joint, since in all other cases the transformation matrix $\mathbf{T}$ is constant and therefore its derivative vanishes.

The expression obtained for the product $\mathbf{T}^{-1} \dot{\mathbf{T}}$ takes the form

$$\mathbf{T}^{-1} \dot{\mathbf{T}} = \tilde{\mathcal{S}}_i \dot{\theta}_i = \begin{bmatrix} \hat{\mathcal{S}}_{i,\text{rot}} & \mathcal{S}_{i,\text{lin}} \\ \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \dot{\theta}_i. \quad (2.66)$$

This expression belongs to the Lie algebra $\mathfrak{se}(3)$ associated with the Euclidean group $SE(3)$. The resulting matrix $\hat{\mathcal{S}}_i$ depends on the type of joint and its corresponding degrees of freedom.

A particular feature of this matrix is that it can be associated with a vector of dimension 6×1 . In the robotics literature, such vectors are commonly referred to as *6D vectors*, while in screw theory they are known as *twists*.

In expression (2.66), two concepts have been introduced implicitly.

The first one is the operator $\widehat{(\cdot)}$, which corresponds to the matrix representation of the vector cross product for $3D$ vectors.

DEFINITION

2.49

Skew-symmetric matrix. This operator $\widehat{(\cdot)}$ is also known as the *cross-product matrix* or *skew-symmetric matrix* (see (2.57)). Its definition for a vector

$$\mathbf{a} = (a_x, a_y, a_z)^\top$$

is given by

$$\hat{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \quad (2.67)$$

The second concept involved is the operator $\widetilde{(\cdot)}$, which transforms a vector belonging to the Lie algebra $\mathfrak{se}(3)$, represented as a 6×1 vector, into a 4×4 matrix representation.

In this context, one may also define the inverse operation, commonly referred to as the anti- $\widetilde{(\cdot)}$ operator, whose purpose is to extract the underlying 6×1 vector from its associated 4×4 matrix representation.

This vector is commonly expressed in the form

$$\mathcal{S}_i = \begin{bmatrix} \mathcal{S}_{i,\text{rot}} \\ \mathcal{S}_{i,\text{lin}} \end{bmatrix}.$$

This representation follows the *Plücker notation*, in which the rotational component is placed first, followed by the linear component.

For each of the classical joint types, the operation

$$\mathbf{T}_{i-1,i}^{-1} \dot{\mathbf{T}}_{i-1,i}$$

can be computed, and the corresponding vector $\mathcal{S}_i$ can subsequently be extracted using the anti- $\widetilde{(\cdot)}$ operation. The resulting vectors, expressed according to the Plücker notation, are summarized below:

- *Revolute joint.*

$$\mathcal{S}_i = [0, 0, 1, 0, 0, 0]^\top.$$

- *Prismatic joint*

$$\mathcal{S}_i = [0, 0, 0, 0, 0, 1]^\top.$$

- *Cylindrical joint*

$$\mathcal{S}_i = \begin{bmatrix} 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 0, 0, 1 \end{bmatrix}^\top.$$

See [62] for additional joint types.

In equation (2.65), it can be observed that each term $\tilde{\mathcal{S}}_i$ appears within an operation of the form $\mathbf{T}^{-1}\tilde{\mathcal{S}}_i\mathbf{T}$. In Lie algebra, the operation

$$\mathbf{T}(\cdot)\mathbf{T}^{-1}$$

is known as the *adjoint operator*.

Although it is formally defined through the inverse transformation,

$$\text{Ad}_{\mathbf{T}_{i-1,i}}\mathcal{S}_i \leftrightarrow \mathbf{T}_{i-1,i}\hat{\mathcal{S}}_i\mathbf{T}_{i-1,i}^{-1},$$

the use of subscript notation together with the adjoint operator property

$$\text{Ad}_{\mathbf{T}_{i,i-1}}^{-1} = \text{Ad}_{\mathbf{T}_{i,i-1}}^{-1} = \text{Ad}_{\mathbf{T}_{i-1,i}},$$

allows the same indexing convention employed for homogeneous transformation matrices to be naturally extended to adjoint operators.

Following this notation,

DEFINITION
2.50

Adjoint operator. The *adjoint operator* is defined as

$$\text{Ad}_{\mathbf{T}_{i-1,i}} = \begin{bmatrix} \mathbf{R}_{i-1,i} & \mathbf{0}_{3 \times 3} \\ \hat{\mathbf{r}}_{i-1,i}\mathbf{R}_{i-1,i} & \mathbf{R}_{i-1,i} \end{bmatrix}. \quad (2.68)$$

On the other hand, Lie algebra introduces the notion of covectors when defining inner products between elements of the Lie algebra. Within the dual space of covectors, there exists an operator analogous to the adjoint operator, known as the *coadjoint operator*.

DEFINITION
2.51

Coadjoint operator. The *coadjoint operator* is defined through the relation

$$\text{Ad}_{\mathbf{T}}^* = \text{Ad}_{\mathbf{T}}^{-\top},$$

which, using the subscript notation, can be written as

$$\text{Ad}_{\mathbf{T}_{i-1,i}}^* = \begin{bmatrix} \mathbf{R}_{i-1,i} & \hat{\mathbf{r}}_{i-1,i} \mathbf{R}_{i-1,i} \\ \mathbf{0}_{3 \times 3} & \mathbf{R}_{i-1,i} \end{bmatrix}. \quad (2.69)$$

Remark 2.24. It is important to note that, according to [63], the *coadjoint operator* is defined as

$$\text{Ad}_{\mathbf{T}}^* = \text{Ad}_{\mathbf{T}^{-1}}^\top.$$

However, it can be verified that, when using the subscript notation adopted in this work, both formulations lead to the same result. $\square$

Using the *adjoint operator* of the Lie algebra, the velocity equation (2.65) for an open kinematic chain can be expressed as

$$\mathbf{V}_n = \sum_{i=1}^n \text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \dot{\theta}_i. \quad (2.70)$$

Furthermore, the product between an adjoint operator and its corresponding coadjoint operator satisfies

$$\text{Ad}_{\mathbf{T}_{i-1,i}}^\top \text{Ad}_{\mathbf{T}_{i-1,i}}^* = \text{Ad}_{\mathbf{T}_{i-1,i}}^* \text{Ad}_{\mathbf{T}_{i-1,i}}^\top = \mathbf{I}.$$

Robotics: Acceleration

The acceleration analysis is obtained by differentiating the velocity analysis. Differentiating equation (2.70) with respect to time yields

$$\mathbf{A}_n = \sum_{i=1}^n \left(\text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \ddot{\theta}_i + \text{Ad}_{\mathbf{T}_{n,i}} \dot{\mathcal{S}}_i \dot{\theta}_i + \text{Ad}_{\mathbf{T}_{n,i}} \text{ad}_{\mathbf{V}_i} \mathcal{S}_i \dot{\theta}_i \right). \quad (2.71)$$

In this expression, the concept of the *Lie bracket operator*, $\text{ad}_{\mathbf{V}}$, has been introduced. This operator can be interpreted as an extension of the vector cross product from $3D$ vectors to $6D$ vectors.

From a mathematical perspective,

DEFINITION 2.52

Lie bracket operator. The *Lie bracket operator* is defined as the result of the following operation between two elements of the Lie algebra:

$$\text{ad}_{\mathbf{V}_i} \mathbf{V}_j = [\widetilde{\mathbf{V}}_i, \widetilde{\mathbf{V}}_j] = \widetilde{\mathbf{V}}_i \widetilde{\mathbf{V}}_j - \widetilde{\mathbf{V}}_j \widetilde{\mathbf{V}}_i, \quad (2.72)$$

for the particular case of matrices belonging to the Lie algebra $\mathfrak{se}(3)$.

However, the same result can also be obtained by differentiating the *adjoint operator* expression given in equation (2.68), namely,

$$\dot{\text{Ad}}_{\mathbf{T}_{i-1,i}} = \text{Ad}_{\mathbf{T}_{i-1,i}} \text{ad}_{\mathbf{V}_i}. \quad (2.73)$$

Remark 2.25. Whether the *Lie bracket operator* is obtained through the differentiation of the adjoint operator, as in (2.73), or through the matrix operations in (2.72), the resulting expression is the same:

$$\text{ad}_{\mathbf{V}_i} = \begin{bmatrix} \hat{\boldsymbol{\omega}}_i & \mathbf{0}_{3 \times 3} \\ \hat{\mathbf{v}}_i & \hat{\boldsymbol{\omega}}_i \end{bmatrix}. \quad (2.74)$$

┘

Remark 2.26. It is important to emphasize that the *Lie bracket operator* possesses properties analogous to those of the vector cross product:

$$\text{ad}_{\mathbf{V}_i} \mathbf{V}_j = -\text{ad}_{\mathbf{V}_j} \mathbf{V}_i, \quad (2.75)$$

$$\text{ad}_{\mathbf{V}_i} \mathbf{V}_i = \text{ad}_{\mathbf{V}_j} \mathbf{V}_j = 0. \quad (2.76)$$

┘

In the acceleration equation (2.71), the term $\dot{\mathcal{S}}_i$ appears. This term only exists for joints in which the vector $\mathcal{S}_i$ is not constant.

Following the previously introduced definitions of $\mathcal{S}_i$, it can be observed that $\mathcal{S}_i$ remains constant for the classical joint types considered here. Consequently, $\dot{\mathcal{S}}_i = 0$. See [62] for the case of more general joint types.

The product $\text{ad}_{\mathbf{V}_i} \mathcal{S}_i \dot{\theta}_i$ which appears in the acceleration equation (2.71), can be commuted using the skew-symmetry property of the Lie bracket operator, yielding the equivalent term $-\text{ad}_{\mathcal{S}_i} \mathbf{V}_i \dot{\theta}_i$.

At this point, substituting equation (2.70) for the computation of $\mathbf{V}_i$ leads to the following expression:

$$\mathbf{A}_n = \sum_{i=1}^n \left(\text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \ddot{\theta}_i + \text{Ad}_{\mathbf{T}_{n,i}} \dot{\mathcal{S}}_i \dot{\theta}_i \right) - \sum_{i=2}^n \text{Ad}_{\mathbf{T}_{n,i}} \text{ad}_{\mathcal{S}_i} \left(\sum_{j=1}^{i-1} \text{Ad}_{\mathbf{T}_{i,j}} \mathcal{S}_j \dot{\theta}_j \right) \dot{\theta}_i. \quad (2.77)$$

The result can be commuted once again in order to change the sign of the expression, obtaining

$$\mathbf{A}_n = \sum_{i=1}^n \left(\text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \ddot{\theta}_i + \text{Ad}_{\mathbf{T}_{n,i}} \dot{\mathcal{S}}_i \dot{\theta}_i \right) + \sum_{i=1}^{n-1} \text{Ad}_{\mathbf{T}_{n,i}} \text{ad}_{\mathcal{S}_i} \left(\sum_{j=i+1}^n \text{Ad}_{\mathbf{T}_{i,j}} \mathcal{S}_j \dot{\theta}_j \right) \dot{\theta}_i.$$

The second summation in the previous equation is referred to as the *Lie acceleration term*, since it arises from the Lie products between different Lie algebra vectors. A

particular characteristic of this term is that only velocity components participate in its evaluation.

By combining the Lie acceleration term with the derivative term

$$\dot{\mathcal{S}}_i,$$

the so-called *Coriolis acceleration terms* are obtained, due to their dependence on the joint velocities:

$$\mathbf{A}_{c,n} = \sum_{i=1}^n \text{Ad}_{\mathbf{T}_{n,i}} \dot{\mathcal{S}}_i \dot{\theta}_i + \sum_{i=1}^{n-1} \text{Ad}_{\mathbf{T}_{n,i}} \text{ad}_{\mathcal{S}_i} \left(\sum_{j=i+1}^n \text{Ad}_{\mathbf{T}_{i,j}} \mathcal{S}_j \dot{\theta}_j \right) \dot{\theta}_i. \quad (2.78)$$

With this definition, the acceleration of a rigid body can be compactly expressed as

$$\mathbf{A}_n = \sum_{i=1}^n \text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \ddot{\theta}_i + \mathbf{A}_{c,n}. \quad (2.79)$$

3 Stability and Optimization of PI Controllers for Time-Delay Systems

Shall I refuse my dinner because I do not fully understand the process of digestion?

Oliver Heaviside

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This chapter addresses the analysis and optimization of proportional–integral controllers for systems with inherent time delays. Stability regions are characterized analytically through the construction of stability crossing boundaries derived from the corresponding quasi-polynomial characteristic equation. The concepts of spectral abscissa and σ -stability are employed to evaluate the exponential decay properties of the closed-loop system. Furthermore, optimization-based methodologies are introduced to determine controller gains that improve transient performance and robustness while preserving closed-loop stability. Numerical examples and simulations are presented to validate the effectiveness of the proposed design procedures.

Time-delay effects are ubiquitous in control systems due to sensing, computation, communication, measurement filtering, actuation, and physical transport phenomena. Although delays are often regarded as undesirable because they deteriorate performance or destabilize feedback loops, they also introduce a rich dynamical structure that demands analytical treatment. Classical PI controllers remain the most widely employed control strategy in industry; however, the presence of a delay fundamentally alters the closed-loop dynamics, transforming the characteristic equation into a quasi-polynomial with infinitely many roots. These complications are well documented in the literature [20] [21].

The analysis and synthesis of PI controllers for systems with delays therefore require specialized mathematical tools that explicitly account for the infinite-dimensional nature of the closed-loop system. Classical methods such as Routh–Hurwitz criteria, Jury tests, or root-locus techniques are no longer directly applicable, since they assume characteristic polynomials of finite degree. For time-delay systems, the characteristic equation involves exponential terms of the form e^{-hs} , which induce infinitely many characteristic roots distributed across the complex plane. The distribution, stability, and sensitivity of these roots depend nontrivially on both the plant parameters and the controller gains.

This chapter develops a unified analytical framework for classical PI control of linear time-delay systems. The results presented here are based on the synergistic methodologies introduced in [12] and extended in [14], where geometric, frequency-domain, and analytical tools are combined to characterize stability regions, determine root-crossing directions, and optimize the spectral abscissa.

The objective of this chapter is twofold. First, we derive explicit analytical conditions that guarantee stability of the closed-loop system under PI control in the presence of a fixed delay. Second, we develop systematic techniques to determine stabilizing parameters and optimize the transient response by manipulating the spectral abscissa. These methodologies rely on the characterization of CRB, RRB, direction-of-crossing roots, and σ -stability analysis. Together, these tools allow the detailed examination of how roots move across the imaginary axis as controller parameters vary, enabling robust and computationally efficient controller synthesis.

3.1 Problem formulation

Consider a LTI SISO system described by

$$G(s) := \frac{P(s)}{Q(s)} e^{-\tau s} = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}e^{-\tau s}, \quad \tau \in \mathbb{R}_+, \quad (3.1)$$

where $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is the state-space representation of the open-loop system. The polynomials $P(s)$ and $Q(s)$ are defined as

$$P(s) := p_m s^m + p_{m-1} s^{m-1} + \dots + p_1 s + p_0, \quad (3.2)$$

$$\begin{aligned} Q(s) &:= q_n s^n + q_{n-1} s^{n-1} + \dots + q_1 s + q_0, \\ &= q_n \prod_{j=1}^n (s - s_j^{(0)}), \end{aligned} \quad (3.3)$$

where $s_j^{(0)} \in \mathbb{C}$, $p_m \neq 0$ and $q_n \neq 0$, which are assumed to satisfy the following conditions.

ASSUMPTIONS

3.1

The polynomials $P(s)$ and $Q(s)$ satisfy the following conditions

- (i) $\deg Q(s) > \deg P(s)$.
- (ii) $P(s)$ and $Q(s)$ are coprime polynomials.
- (iii) $|P(i\omega)| > 0, \forall \omega \in \mathbb{R}$.

Remark 3.1. Assumption 3.1-(i) states that the system is causal. In the case that Assumption 3.1-(ii) is not fulfilled, it would imply the existence of a non-constant common factor $c(s)$, such that $P(s) = c(s)\tilde{P}(s)$ and $Q(s) = c(s)\tilde{Q}(s)$. Under such a situation, choosing $c(s)$ to be of the highest possible degree, the analysis could be pursued if it $c(s)$ is a Hurwitz polynomial; otherwise, the system will remain unstable independently of the control action. Finally, to simplify the presentation, (iii) avoids having roots on the imaginary axis in P . $\lrcorner$

The problems considered in this chapter can be summarized as follows.

Problem 3.1: Stability conditions

Determine analytical conditions on the parameters (k_p, k_i) of the PI controller

$$C(s) = k_p + \frac{k_i}{s}, \quad (3.4)$$

such that the closed-loop system, consisting of (3.1), the PI controller in (3.4) and a unitary feedback (as shown in Figure 3.1), is asymptotically stable in a regulation task.

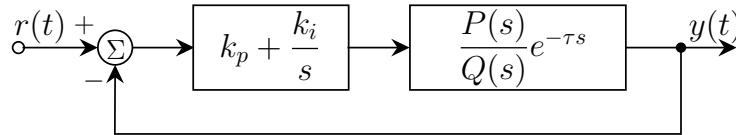


Figure 3.1 / Schematic representation of a conventional closed-loop control system.

Problem 3.2: Optimization performance indices

Given a cost function J , composed of one or more performance indices listed in Table 3.1, the goal is to solve the following optimization problem

$$\mathbf{k}^* = \underset{\mathbf{k} \in \mathcal{G}_l}{\operatorname{argmin}} \{J(\mathbf{k})\} \quad (3.5)$$

where $\mathbf{k} := [k_p, k_i]^\top$, and $\mathcal{G}_l$ denotes the feasible set for the controller gains k_p and k_i . This set will correspond to a discretization of the region containing all stabilizing controllers.

Given our interest in identifying stability regions within the (k_p, k_i) parameter

Performance index	Symbol	Expression
Settling time	$t_s(\alpha)$	$ r(t) - y(t) \leq \alpha, \forall t \geq t_s$
Control Effort Square (CES)	CES	$\int_0^t u(\tau)^2 d\tau$
Control Variation Square (CVS)	CVS	$\int_0^t \dot{u}(\tau)^2 d\tau$
Integral Time Absolute Error (ITAE)	ITAE	$\int_0^t \tau e(\tau) d\tau$
Integral Time Squared Error (ITSE)	ITSE	$\int_0^t \tau e(\tau)^2 d\tau$

Table 3.1 / Index of transient-state performance data.

space of the closed-loop system, analyzing the roots of its characteristic function becomes necessary. This function is defined as

$$\Delta(s; k_p, k_i) := sQ(s) + (k_p s + k_i)P(s)e^{-\tau s}. \quad (3.6)$$

Equation (3.6) is a *retarded quasi-polynomial*, that is, a characteristic function consisting of a polynomial part and a delayed exponential. Unlike polynomials, quasi-polynomials generate infinitely many roots; however, these roots depend continuously on (k_p, k_i, h) [22] [49]. The key question is determining the parameter values for which all roots lie strictly in the open LHP.

The presence of the exponential term e^{-hs} creates characteristic roots that are not constrained to finite algebraic multiplicity, producing a rich geometry in the root distribution. Analyzing these distributions requires specialized tools, which are developed in the following sections.

Remark 3.2. The roots of quasipolynomials are continuous functions with respect to their parameters (see, for instance, [22] [64], and related sources). Consequently, the number of roots in the RHP can vary only if some cross the imaginary axis. $\lrcorner$

Following this, some definitions will be introduced, which are fundamental for presenting our results.

DEFINITION
3.1

The *frequency crossing set* $\Omega \subset \mathbb{R}$ is the set of all ω (crossing frequency^a) for which there exists at least a pair $(k_p, k_i) \in \mathbb{R}^2$ satisfying

$$\Delta(i\omega; k_p, k_i) = i\omega Q(i\omega) + (i\omega k_p + k_i)P(i\omega)e^{-i\omega\tau} = 0. \quad (3.7)$$

^aAlso called singular frequency.

Remark 3.3. Since Q and P are polynomials with real coefficients, it is clear to see that the following relation holds

$$\Delta(-i\omega; k_p, k_i) = \overline{\Delta(i\omega; k_p, k_i)}. \quad (3.8) \quad \lrcorner$$

◁ In other words, the above equality says that the frequency crossing set can be reduced to the set of nonnegative real numbers, that is, $\Omega \subset \overline{\mathbb{R}}_+$. ▢

DEFINITION

3.2

The *stability crossing curves*^a will be denoted by $\mathcal{T}$, and is defined as the set of all parameters $(k_p, k_i) \in \mathbb{R}^2$ for which there exists at least one $\omega \in \Omega$ satisfying $\Delta(i\omega; k_p, k_i) = 0$. Moreover, any point $\mathbf{k} \in \mathcal{T}$ is known as a *crossing point*.

^aAlso called stability boundaries.

It should be noted that when the quasi-polynomial (3.6) is stable, all of its roots lie in the LHP. These roots can only transition to instability under two conditions (see Figure 3.7): (1) through the origin of the complex plane, known as the RRB, or (2) a set of singular frequencies called CRB. These boundaries collectively define the closed-loop stability map, denoted as $\mathcal{S}$.

3.2 Analytical Approach

A key concept in determining stability regions is the *continuity root property* of quasi-polynomials. This property, inherent to quasi-polynomials of the retarded type¹, can be formally stated as follows.

PROPOSITION

3.1

For a given $\mathbf{k}^* := [k_p^*, k_i^*]^\top \in \mathbb{R}^2$ and $\tau^* > 0$, let $s^* \in \mathbb{C}$ be a characteristic root of (3.6) with multiplicity $\ell \in \mathbb{N}$. Then, there exists a constant $\bar{\varepsilon} > 0$, such that $\forall \varepsilon \in (0, \bar{\varepsilon})$, there is a number $\delta > 0$ such that $\Delta(s; \mathbf{k})$, where $\|\mathbf{k} - \mathbf{k}^*\|_2 < \delta$ and $0 < |\tau - \tau^*| < \delta$, has exactly ℓ -roots in the disk $\{s \in \mathbb{C} : |s - s^*| < \varepsilon\}$.

Remark 3.4. The above property holds for systems of the retarded type, which, due to Assumption 3.1-(i), is the case considered in this work. ▢

Strictly speaking, the above result states that the characteristic roots behave continuously with respect to continuous variations of system parameters and delay τ .

PROPOSITION

Let $\tau \in \mathbb{R}_+$ be a fixed value. Then, $\omega \in \Omega$ is a *crossing*

¹Quasi-polynomials of the retarded type arise when the highest-order derivative term does not include a time delay.

frequency if and only if $\mathbf{k}(\omega) := [k_p(\omega), k_i(\omega)]^\top$, where

$$k_p(\omega) := -\Re\left(\frac{Q(i\omega)}{P(i\omega)} e^{i\omega\tau}\right), \quad (3.9)$$

$$k_i(\omega) := \omega \Im\left(\frac{Q(i\omega)}{P(i\omega)} e^{i\omega\tau}\right), \quad (3.10)$$

defines a crossing point $\mathbf{k}(\omega) \in \mathcal{T}_\omega$.

Proof. Consider the characteristic equation (3.6). According to Definition 3.2, all the crossing points $\mathbf{k} \in \mathcal{T}_\omega$ are composed of all pairs (k_p, k_i) that solve equation (3.7) for $s = i\omega$. Hence, by taking the real and imaginary parts to the equation $\Delta(i\omega; k_p, k_i) = 0$, yields to

$$\Re(\Delta(i\omega; k_p, k_i)) = 0, \quad (3.11)$$

$$\Im(\Delta(i\omega; k_p, k_i)) = 0. \quad (3.12)$$

By solving (3.11) and (3.12) for k_p and k_i leads, by simple algebraic manipulations, to equations (3.9) and (3.10). $\square$

PROPOSITION
3.3

Let $\tau \in \mathbb{R}_+$ be a fixed value. Then, the line $\mathbf{k}_0 = [k_{p0}, k_{i0}]^\top \in \mathbb{R}^2$ defined by

$$\mathbf{k}_0 := \left\{ \begin{bmatrix} k_{p0} \\ 0 \end{bmatrix} \middle| k_{p0} \in \mathbb{R} \right\}, \quad (3.13)$$

defines a *stability crossing curve*. Furthermore, this characterizes a crossing through the origin of the complex plane.

Proof. By taking $s = 0$ in (3.6), the result is obtained as follows

$$\Delta(0; k_p, k_i) = 0,$$

$$k_i P(0) = 0.$$

By the Assumption 3.1-(iii), $P(0) \neq 0$. Therefore, from the last expression, the result follows straightforwardly. $\square$

Remark 3.5. $\mathcal{T}_0$ is the set of all gains $\mathbf{k}$, such that equation (3.6) has at least a solution on the horizontal line defined in (3.13). $\lrcorner$

Remark 3.6. The complete stability boundary is the union:

$$\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_\omega.$$

$\lrcorner$

Example 3.1: Second Order System

Let the system be defined by

$$G(s) = \frac{s+1}{s^2+2s+3} e^{-0.3s} \quad (3.14)$$

then the regions of stability are obtained using the Propositions 3.2-3.3

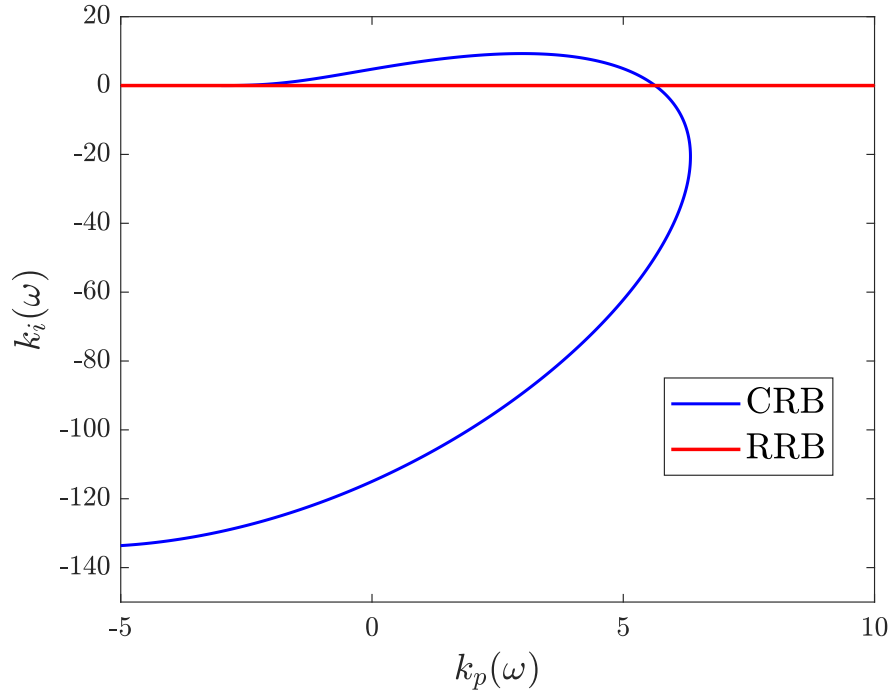


Figure 3.3 / Stability crossing curves for system (3.14).

In the example shown above, it is easy to identify the stability region since the CRB is clearly increasing and does not intersect itself. However, there are also systems where this does happen, i.e., the CRB curve intersects itself and subregions are generated, which makes it difficult to identify the stability region.

Example 3.2: Third Order System

Let the system be defined by

$$G(s) = \frac{s-1}{s^3+5s^2+2s+3} e^{-4s} \quad (3.15)$$

then the regions of stability are obtained using the Propositions 3.2-3.3

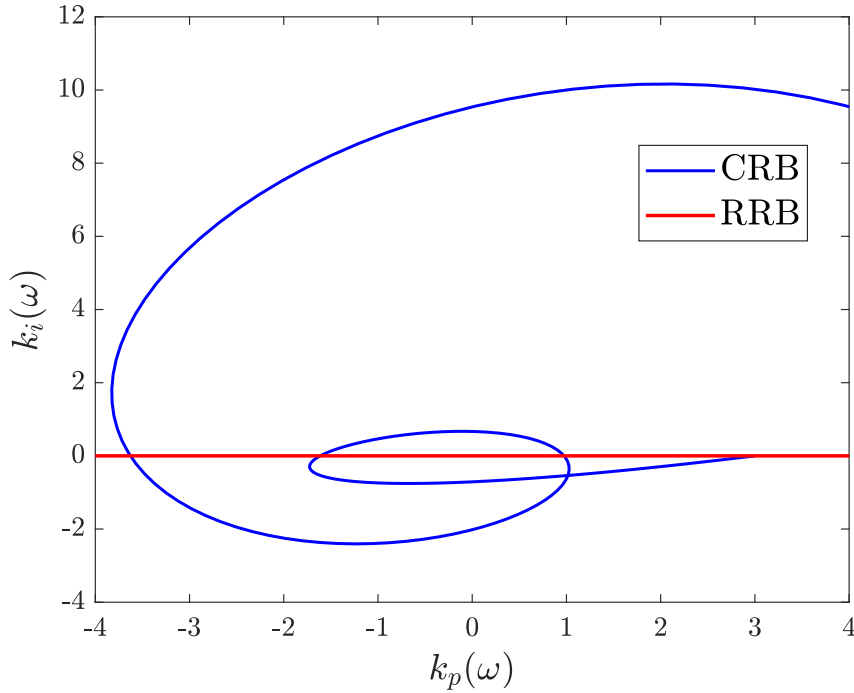


Figure 3.4 / Stability crossing curves for system (3.15).

The following result provides a method for identifying unstable regions. Specifically, any region intersecting the curve $\mathcal{T}_u$ corresponds to an unstable region.

PROPOSITION
3.4

Assume that $|P(x)| > 0, \forall x > 0$. Let $\mathcal{T}_u \in \mathbb{R}^2$ be a curve $\mathcal{T}_u = [k_{pu}(x), k_{iu}(x)]^\top$ defined by

$$\mathcal{T}_u := \begin{bmatrix} e^{\tau x} \frac{xQ(x)P'(x) - P(x)[Q(x) + \tau xQ(x) + xQ'(x)]}{P(x)^2} \\ e^{\tau x} x^2 \frac{-Q(x)P'(x) - P(x)[\tau Q(x) + Q'(x)]}{P(x)^2} \end{bmatrix}, \quad (3.16)$$

where $x \in \mathbb{R}_+$, P' and Q' are the derivatives of P and Q , respectively. Then, all regions of the controller parameter space intersecting $\mathcal{T}_u$ are *unstable*.

Proof. The proof can be demonstrated by observing that all points on $\mathcal{T}_u$ correspond to numerous positive real roots. Then, by solving $\Delta(x; k_p, k_i) = 0$ and $\frac{d}{dx} \Delta(x; k_p, k_i) = 0$ with respect to k_{pu} and k_{iu} , the curve (3.16) is obtained directly. $\square$

It is worth noting that Propositions 3.2–3.3 establish the stability crossing curves, which partition the parameter plane into regions where the number of unstable roots remains invariant. In other words, each region contains the same number of roots in the RHP of the complex plane.

In this context, according to Proposition 3.1, if a continuous path in the parameter space is taken from an unstable region ($\mathcal{R}_2$) to a stable region ($\mathcal{R}_1$), at least one root must continuously transition from the RHP to the LHP, as illustrated in Figure 3.5. A similar observation holds when the transition occurs in the opposite direction, from $\mathcal{R}_1$ to $\mathcal{R}_2$.

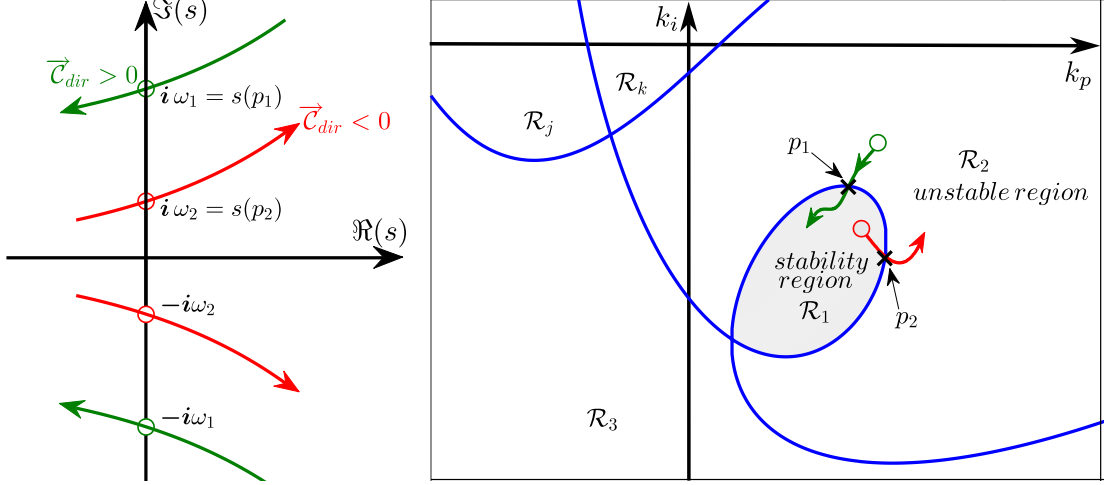


Figure 3.5 / Behavior of root crossings in a quasi-polynomial.

Hence, it will be of core importance to propose tools that will allow discriminating the stability regions, i.e., results that will allow us to find regions having only stabilizing pairs (k_p, k_i) . For such an end, the following result allows computing the crossing of roots from the RHP to the LHP.

PROPOSITION
3.5

Let $\mathbf{k}^* := [k_p^*, k_i^*]^\top \in \mathbb{R}^2$ be a critical gain for the *crossing frequency* $\omega^* \in \mathbb{R}_+$, and assume that $|Q(i\omega)| > 0, \forall \omega \in \mathbb{R}$. Then, a pair of simple roots of the closed-loop quasi-polynomial (3.6) will cross from the LHP to the RHP as $\mathbf{k} := [k_p, k_i]^\top$ traverse the CRB ($\mathcal{T}_\omega$) at $\mathbf{k}^*$ in the increasing direction of k_p if

$$\vec{\mathcal{C}}_{dir}(\omega^*, \mathbf{k}^*) < 0, \quad (3.17)$$

where $\vec{\mathcal{C}}_{dir}$ is defined as

$$\vec{\mathcal{C}}_{dir}(\omega, \mathbf{k}) := \Re \left\{ k_i - i\omega P'(i\omega) + i\omega(k_p i\omega + k_i)(\tau + \varphi(\omega)) \right\},$$

with $P'(i\omega) := \frac{dP(s)}{ds} \Big|_{s=i\omega}$, and $\varphi(\omega) := \sum_{j=1}^n \frac{1}{i\omega - s_j^{(0)}}$,

where the term $s_j^{(0)}$ is defined in (3.3). Furthermore, the crossing is from the RHP to the LHP if the inequality (3.17) is reversed (see Figure 3.5).

Proof. By assumption, $s^* = i\omega$ (with $\omega > 0$) is a simple root of Δ for $\mathbf{k}$. Additionally, since by hypothesis $|Q(i\omega)| > 0$, it follows that φ is well-defined in Ω . Consequently, any solution $s(\mathbf{k}^*)$ will cross the imaginary axis from the LHP to the RHP (or from the RHP to the LHP) via a continuous complex path. Thus, bearing in mind such an observation while assuming that $\mathbf{k}$ will cross $\mathcal{T}$ in the increasing direction of k_p , the crossing will be from the LHP to the RHP (or from RHP to LHP) if the following inequality holds

$$\Re\left\{\frac{ds}{dk_p}\right\} > 0, \quad (< 0).$$

Now, by considering the *implicit function theorem* (see, for instance, [65]), it follows that

$$\frac{ds}{dk_p} = - \frac{\left. \frac{\partial \Delta(s^*; \mathbf{k})}{\partial k_p} \right|_{\mathbf{k}=\mathbf{k}^*}}{\left. \frac{\partial \Delta(s; \mathbf{k}^*)}{\partial s} \right|_{s=s^*}}.$$

Therefore, the proof is complete by noticing that

$$\text{sign}\left\{\Re\left\{\left.\frac{\partial \Delta(s^*; \mathbf{k})}{\partial k_p}\right|_{\mathbf{k}=\mathbf{k}^*}\right\}\right\} \equiv \text{sign}\{\vec{\mathcal{C}}_{dir}(\omega^*, \mathbf{k}^*)\}.$$

□

Remark 3.7. The arrow notation $\vec{(\cdot)}$ over the symbol $\mathcal{C}_{dir}$ is used as an annotation to indicate that a path in the parameter space crosses the stability boundary at $\mathbf{k}^*$. Thus, $\vec{\mathcal{C}}_{dir}$ specifically denotes the direction in which the solutions traverse the stability boundary in the complex plane. ┘

COROLLARY

3.1

Let $\mathbf{k}^* := \mathbf{k}_0 \in \mathbb{R}^2$, and assume that $|Q(0)| > 0$. Then, one root of the closed-loop quasi-polynomial (3.6) will *cross* from the LHP to the RHP through the origin as $\mathbf{k} := [k_p, k_i]^\top$ traverse the RRB ($\mathcal{T}_0$) at $\mathbf{k}^*$ in the increasing direction of k_i if

$$\frac{p_0}{q_0 + p_0 k_p^*} < 0. \quad (3.18)$$

Furthermore, the crossing is from the RHP to the LHP if the inequality (3.18) is reversed.

Given $\sigma \in \mathbb{R}_+$, the σ -*stability problem* consists of determining the controller gains $\mathbf{k}$ such that the real part of the rightmost roots of the characteristic function for the closed-loop system is less than or equal to $-\sigma$. To fully describe these control gains, denote $\mathcal{T}_\sigma$ as the set of all gains $\mathbf{k}$ such that equation (3.6) has at least one solution

on the vertical line in the complex plane defined as $\mathbf{L}_\sigma := \{s \in \mathbb{C} : \Re[s] = -\sigma\}$ (also referred to as σ -axis). Define Ω_σ as the corresponding frequency set

$$\Omega_\sigma := \{\omega \in \mathbb{R}_+ : P(-\sigma + i\omega) \neq 0\},$$

and consider the following results.

PROPOSITION
3.6

Let $\omega \in \Omega_\sigma$, and let $\sigma \in \mathbb{R}_+$ be fixed value. Then, the set $\mathcal{T}_\sigma$ is given by

$$\mathcal{T}_\sigma = \mathcal{T}_{\bar{\sigma}} \cup \mathcal{T}_{\sigma,0}, \quad (3.19)$$

with

$$\mathcal{T}_{\sigma,0} = \left\{ \mathbf{k} \mid k_i = \sigma \left[k_p + \frac{Q(-\sigma)}{P(-\sigma)} e^{-\tau\sigma} \right] \right\},$$

$$\mathcal{T}_{\bar{\sigma}} = \{ \mathbf{k} \mid \mathbf{k} = \mathbf{k}_\sigma(\omega) \},$$

where $\mathbf{k}_\sigma(\omega) := [k_{p\sigma}(\omega) \ k_{i\sigma}(\omega)]^\top$, with gains $k_{p\sigma}$ and $k_{i\sigma}$ defined by

$$k_{p\sigma}(\omega) = \frac{e^{-\tau\sigma}}{\omega} \left[\sigma \Im \left(\frac{Q(\xi)}{P(\xi)} e^{i\tau\omega} \right) - \omega \Re \left(\frac{Q(\xi)}{P(\xi)} e^{i\tau\omega} \right) \right], \quad (3.20a)$$

$$k_{i\sigma}(\omega) = \frac{(\sigma^2 + \omega^2)e^{-\tau\sigma}}{\omega} \Im \left(\frac{Q(\xi)}{P(\xi)} e^{i\tau\omega} \right), \quad (3.20b)$$

where $\xi := -\sigma + i\omega$.

Proof. By substituting $s = -\sigma + i\omega$ into (3.6), the proof is completed by following similar steps to those presented in the proof of Proposition 3.2. $\square$

PROPOSITION
3.7

Let $\mathbf{k}_\sigma(\omega)$ be defined by (3.20). Then, the point of departure $\mathbf{k}_{\sigma,0}$ for the curve $\mathbf{k}_\sigma$ is given by

$$\mathbf{k}_{\sigma,0} := e^{-\tau\sigma} \begin{bmatrix} -\frac{Q(-\sigma)}{P(-\sigma)} \\ \sigma^2 \frac{(Q'(-\sigma) + \tau Q(-\sigma))P(-\sigma) - Q(-\sigma)P'(-\sigma)}{P^2(-\sigma)} \end{bmatrix}. \quad (3.21)$$

Proof. The proof follows straightforwardly by computing

$$\mathbf{k}_{\sigma,0} \equiv \lim_{\omega \rightarrow 0} \mathbf{k}_\sigma(\omega).$$

$\square$

3.3 Metaheuristic–Analytical Approach

After determining the stability region of the closed-loop system, the next step is to identify the controller parameter values that minimize the cost function J . Since an analytical solution is challenging, we propose a “*synergistic approach*”, as described in Algorithm 1.

Algorithm 1 Synergistic optimization process for selecting controller gains.

Require: The transfer function $G(s)$, maximum frequency ω_f , cost function J ;

Ensure: The optimal control $\mathbf{k}^* := [k_p^*, k_i^*]^T$;

```

1: Data:  $\omega_0 \leftarrow 0$ ;
2: Define a strictly increasing finite sequence  $(\omega_\kappa)$  in  $[\omega_0, \omega_f]$ ;
3: Define  $\mathcal{T}_\omega$  and  $\mathcal{T}_0$ ;
4:  $\ell \leftarrow N$ ; ▷ Number of  $\ell$ -regions
5: if  $\mathcal{T}_\omega$  self-intersects then
6:   Find  $\omega_1$  and  $\omega_2$ , such that  $\omega_0 \leq \omega_1 < \omega_2 \leq \omega_f$ ;
7:   Redefine  $\Omega \leftarrow [\omega_1, \omega_2]$ ;
8:   Generate  $k_p(\omega)$  and  $k_i(\omega)$  for  $\omega \in \Omega$ ;
9:    $b \leftarrow \frac{1}{\ell}(\max(k_p(\omega)) - \min(k_p(\omega)))$ ; ▷ rectangle base
10: else
11:    $b \leftarrow \frac{1}{\ell}(k_p(\omega_f) - k_p(\omega_0))$ ; ▷ rectangle base
12: end if
13:  $j \leftarrow 1$ ;
14: while  $j \leq \ell$  do
15:   if  $\mathcal{T}$  self-intersects then
16:      $k_p^l \leftarrow \min(k_p(\omega)) + (j-1)b$ ;
17:      $k_p^u \leftarrow \min(k_p(\omega)) + jb$ ;
18:   else
19:      $k_p^l \leftarrow k_p(\omega_0) + (j-1)b$ ;
20:      $k_p^u \leftarrow k_p(\omega_0) + jb$ ;
21:   end if
22:    $p_r \leftarrow ((k_p(\omega) \geq k_p^l) \wedge (k_p(\omega) \leq k_p^u))$ ;
23:    $k_{i_r} \leftarrow k_i(p_r)$ ;
24:    $k_i^l \leftarrow \min(k_{i_r})$ ;  $k_i^u \leftarrow \max(k_{i_r})$ ;
25:   Select optimization method;
26:   if opt == 'fmincon' then
27:      $k_p^0 \leftarrow \frac{1}{2}(k_p^l + k_p^u)$ ;
28:      $k_i^0 \leftarrow \frac{1}{2}(k_i^l + k_i^u)$ ;
29:   end if
30:   Compute:
   minimize  $J$ 
   subject to  $\mathbf{k} \in \mathcal{G}_\ell$ ;
31:    $j \leftarrow j + 1$ ;
32: end while
33: Find the minimum of all  $J_\ell^l$  obtained to select  $\mathbf{k}^*$ ;

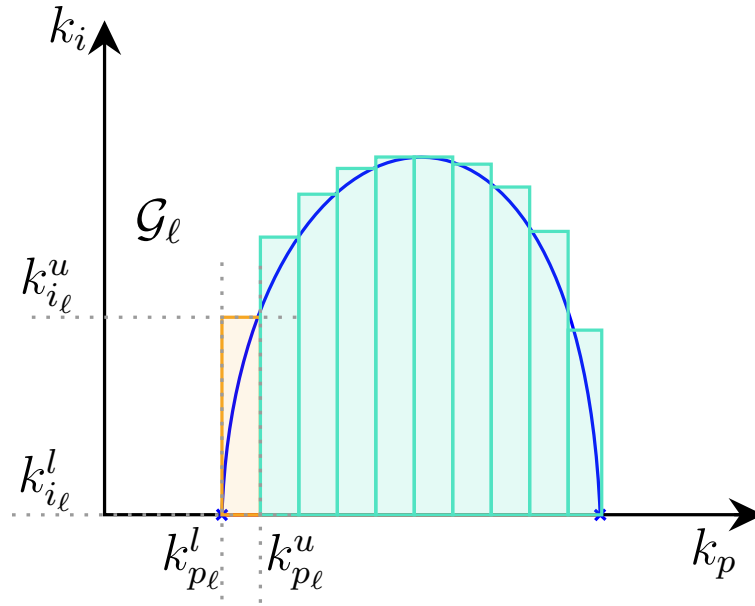
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The proposed algorithm begins by analytically computing the stability region $\mathcal{S}$ in terms of the $\mathcal{T}_\omega$ and $\mathcal{T}_0$ curves (see Figures 3.6a or 3.6b), such that

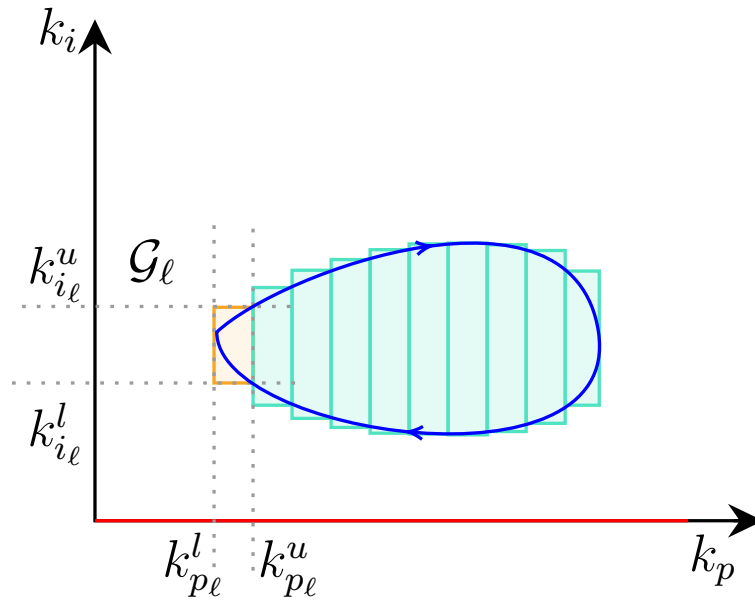
$$\partial\mathcal{S} \subset \mathcal{T}_\omega \cup \mathcal{T}_0, \quad (3.22)$$

with $\partial\mathcal{S}$ denoting the boundary of the stability region.

Next, given a natural number $n \in \mathbb{N}$, the algorithm discretizes the stability region $\mathcal{S}$ into n subregions $\mathcal{G}_\ell$, such that $\mathcal{S} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2 \cup \dots \cup \mathcal{G}_n$. This process requires



(a) Discretization of region $\mathcal{G}$: case (i).



(b) Discretization of region $\mathcal{G}$: case (ii).

Figure 3.7 / Transformation of the stability crossing curves from the complex plane into the parameter space.

detecting whether the $\mathcal{T}_\omega$ curve is self-intersecting and can lead to the following two cases:

- (i) Suppose the curve $\mathcal{T}_\omega$ is simple and not self-intersecting (see Figure 3.6a). In that case, it is easy to discretize since the straight line $\mathcal{T}_0$ helps to compute the bases of the discretization rectangles (see Figure 3.6a), which are all collinear.
- (ii) If $\mathcal{T}_\omega$ is self-intersecting (see Figure 3.6b), the process changes since the stability region does not intersect the $\mathcal{T}_0$ curve. Consequently, the bases of

the rectangles must be calculated individually (see Figure 3.6b).

In other words, we desire to determine the regions $\mathcal{G}_\ell$, where $\ell \in \mathbb{N}$ and $\ell \geq 1$ as search subspaces. $\mathcal{G}_\ell$ is a rectangle with a base and a height; these two variables are defined in detail in Algorithm 1, between lines 15-21. The subspaces $\mathcal{G}_\ell$ are defined as

$$\mathcal{G}_\ell := \left\{ (k_p, k_i) \mid k_{p_\ell}^l \leq k_p \leq k_{p_\ell}^u \wedge k_{i_\ell}^l \leq k_i \leq k_{i_\ell}^u \right\}. \quad (3.23)$$

Again, regions $\mathcal{G}_\ell$ result from discretizing the $\mathcal{S}$ region and aim to reduce the search space to compact areas, thereby ensuring a reduction in processing time. Figures 3.6a and 3.6b show these regions.

Remark 3.8. If $\ell = 1$, the resulting region is a single regular quadrilateral that fully encloses the entire stability region $\mathcal{S}$, i.e., $\mathcal{G} \supseteq \mathcal{S}$. Furthermore, for $\ell = 1, 2, \dots, N$, the following inclusion holds:

$$\bigcup_{\ell=1}^N \mathcal{G}_\ell \supseteq \mathcal{S}. \quad (3.24)$$

Thus, the set of stabilizing gains within the stability crossing curves belongs to the discretized region and, consequently, defines the search space for the optimization algorithm. In this framework, the stability region $\mathcal{S}$ is constructed analytically, while $\mathcal{G}$ is formed as the union of all $\mathcal{G}_\ell$. Since $\mathcal{S}$ is a compact set, this property is inherited by the sets $\mathcal{G}_\ell$ and, consequently, by $\mathcal{G}$. Moreover, given that the cost function J under consideration is convex, a direct application of the Weierstrass theorem (see, e.g., [66]) guarantees that J attains its minimum in $\mathcal{S}$, and therefore in $\mathcal{G}$.

Once the search space is determined, the next step is to select an optimization method. In this work, we employ both the genetic algorithm (GA) and the interior-point algorithm.

Remark 3.9. The optimization process utilizes MATLAB functions from the Optimization Toolbox. The genetic algorithm is implemented using the function `ga`, while constrained optimization (interior-point) is handled by the function `fmincon`.

The GA progressively updates a population of candidate solutions over successive iterations. At each generation, a subset of individuals is randomly chosen from the current population to serve as parents, and new offspring are generated from them to form the subsequent population. After several successive generations, the population evolves towards an optimal solution [67].

On the other hand, the interior-point algorithm facilitates the minimization of a constrained nonlinear multivariable function. Unlike GA, this method generates a single point at each iteration; it approximates an optimal solution with a sequence of points, and a deterministic calculation selects the next point. This method

requires initial conditions; for this, the initial conditions are chosen to correspond to the centroid of each subregion $\mathcal{G}_\ell$.

Therefore, the algorithm follows an iterative approach and performs ℓ -optimizations. These optimizations aim to identify the ℓ -th minimum of the cost function (J_ℓ^l) for each $\mathcal{G}_\ell$ subspace. Ultimately, the algorithm determines the minimal value among these minimums to select the optimal gains ($\mathbf{k}^*$).

Remark 3.10. The proposed methodology employs GA and the interior-point method but remains flexible. Other optimization techniques, such as particle swarm optimization or gradient descent, can also be applied within this synergistic analytical–metaheuristic framework. $\square$

Note 3.1: Experimental hardware

All numerical simulations and experimental evaluations were performed using the same computational platform. The computer operated under Windows 11 and was equipped with an AMD Ryzen 7 5800X CPU, an NVIDIA GeForce RTX 3060 GPU, 16 GB of RAM, and a 931.5 GB solid-state drive. MATLAB R2023b was used to conduct the simulation studies.

Both optimization functions—`ga` and `fmincon`—were configured with specific parameters customized to the problem at hand.

Note 3.2: ga configurations

For the genetic algorithm (`ga`), the population size was set to 200, the number of generations to 10, and the constraint tolerance to 1×10^{-15} . To ensure reproducibility while maintaining diversity in the solution space, we initialized the random number generator using `rng(1, 'twister')`. All other parameters were left at their default values.

Note 3.3: fmincon configurations

For the constrained optimization function (`fmincon`), we selected the “interior-point” algorithm and set the maximum number of iterations to 100. The optimality and constraint tolerances were specified as 1×10^{-15} , while all other settings were retained at their default values.

3.4 Numerical Examples

Example 3.3: Unstable open-loop system (Second-order system)

Consider an unstable second-order system with non-minimum-phase dynamics, represented by the following transfer function

$$G_1(s) = \frac{s - 2}{s^2 - \frac{1}{2}s + \frac{13}{4}} e^{-2s}. \quad (3.25)$$

The first step involves determining the set of invariant regions $\mathcal{R}_k$, where $k \in \bar{\mathbb{N}}$ represents the number of unstable roots. This is achieved by applying Propositions 3.2 and 3.3. The resulting stability crossing curves are depicted in Figure 3.8.

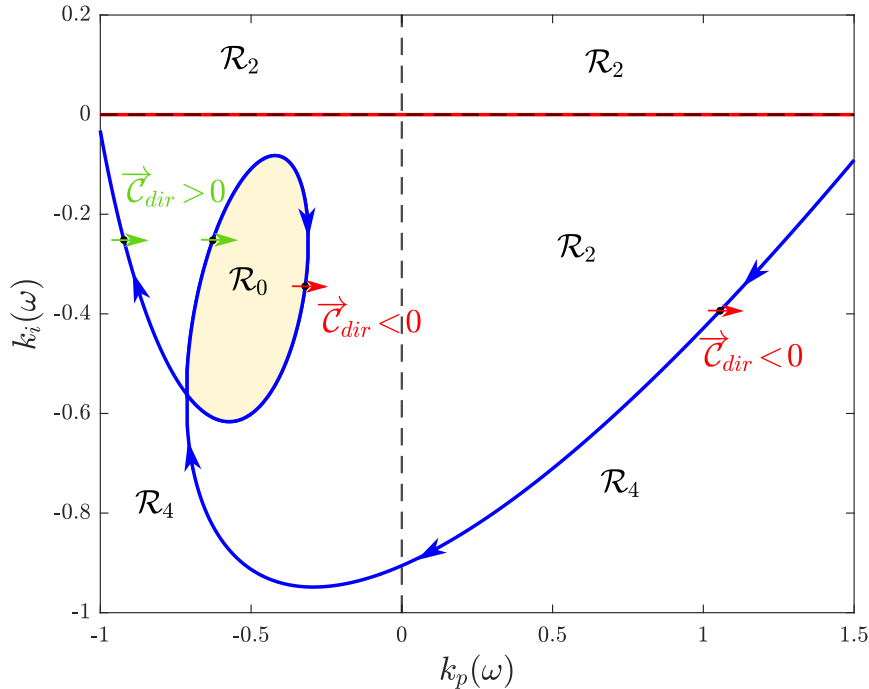


Figure 3.8 / Direction of root crossings in the parameter space.

Next, to identify the stable regions, we apply Proposition 3.5 in conjunction with Corollary 3.1, resulting in the data summarized in Table 3.2.

Singular frequency	Controller gains		Cross direction
ω^*	k_p^*	k_i^*	$\text{sign}\{\vec{\mathcal{C}}_{dir}(\omega^*, \mathbf{k}^*)\}$
0.35	1.0554	-0.3938	—
1.35	-0.6271	-0.2522	+
1.9	-0.32205	-0.3445	—
2.43	-0.9219	-0.2515	+

Table 3.2 / Numerical values of the crossing directions of the system (3.25).

Figure 3.8 displays the enclosed region formed by the self-intersection, which corresponds to the stable region ($\mathcal{R}_0$). Since the stability region in Figure 3.8 has a non-homogeneous shape, meaning it is self-intersecting, we apply the steps of Algorithm 1 to discretize the stable region into 50 sub-regions ($\ell = 50$), as illustrated in Figure 3.9

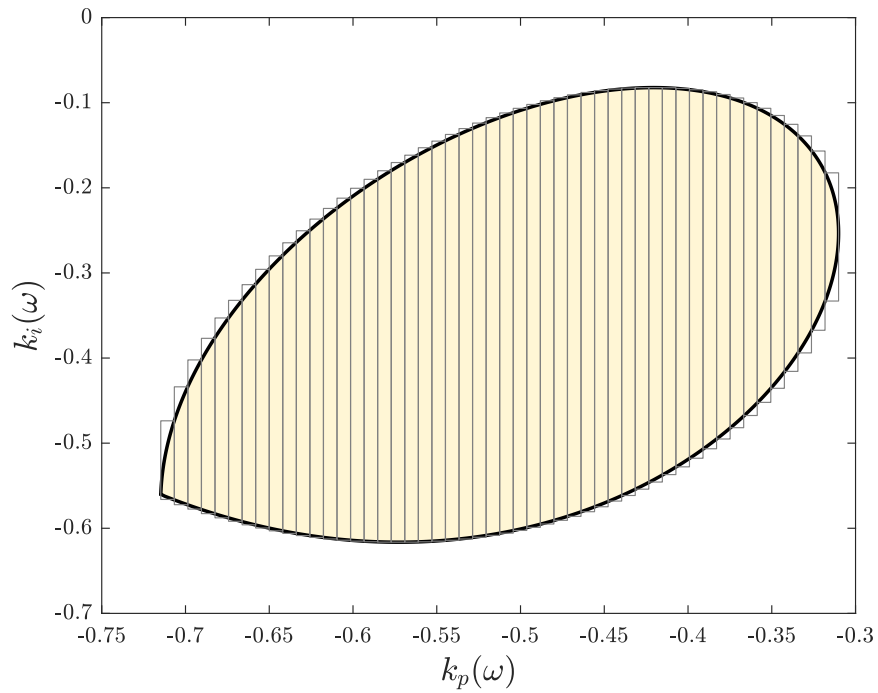


Figure 3.9 / Discretization of the stability region for controller tuning.

In this example, our objective is to minimize the settling time t_s as much as possible. Thus, the cost function is defined as:

$$J_1 = t_s(\alpha)^2, \quad (3.26)$$

where $\alpha = 0.01$.

Figure 3.10 illustrates that minimizing the cost function J_1 using `ga` or `fmincon` results in control gains that closely match those obtained via the MED analytic approach [8] [9].

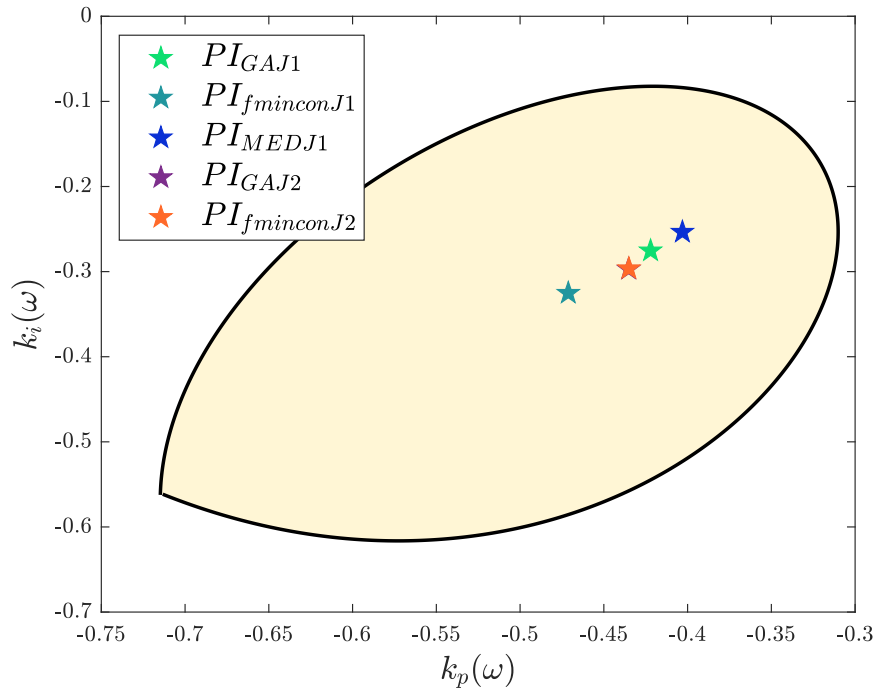
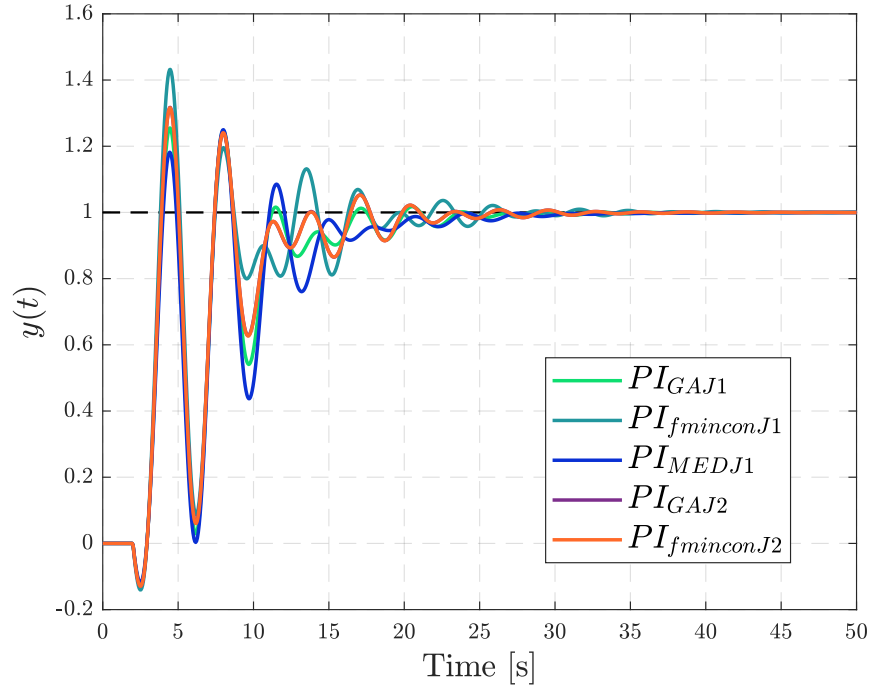
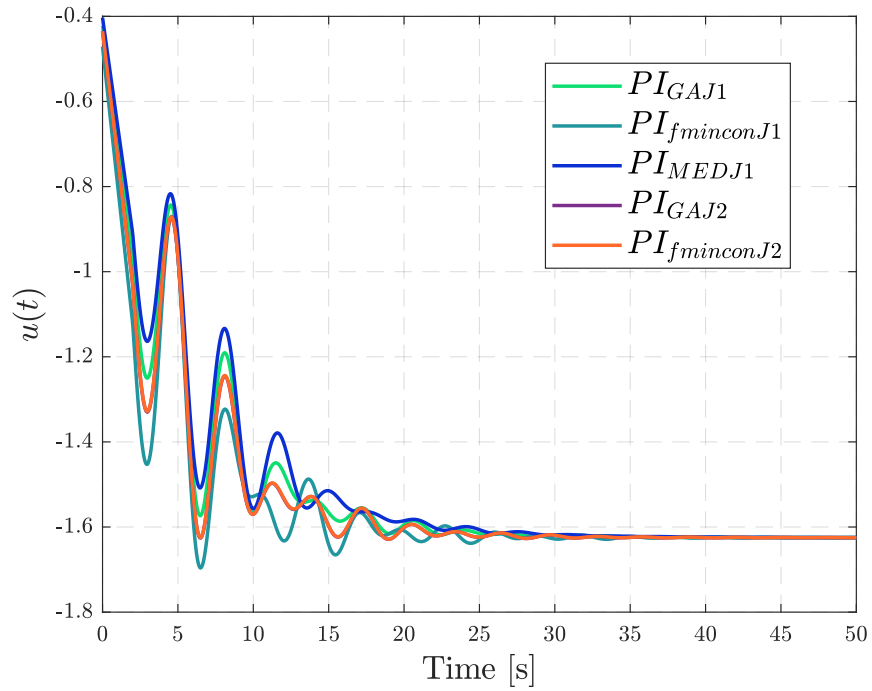


Figure 3.10 / Analysis of the (k_p, k_i) stability region for the system (3.25).

Furthermore, the system response (Figure 3.11a) and control effort (Figure 3.11b) exhibit similar time-domain behavior.



(a) Closed-loop time response of the system (3.25) with distinct controller gains. Controller gain locations are shown in Figure 3.10.



(b) Control efforts corresponding to the responses shown in Figure 3.11a.

Figure 3.12 / Total time responses of system (3.25) with a step-type entry optimizing J_1 and J_2 , with $\ell = 50$.

The **ga** algorithm produced the gains $(k_p^*, k_i^*) = (-0.4221, -0.2754)$, achieving a settling time of $t_s = 26.0146$ s. The **fmincon** optimizer yielded the controller $(k_p^*, k_i^*) = (-0.4232, -0.2735)$ with a settling time of $t_s = 26.0203$ s. Finally, the MED design methodology resulted in the controller $(k_p, k_i) = (-0.4031, -0.2535)$, which led to a settling time of $t_s = 29.7050$ s.

Based on these results, it is evident that the **ga** algorithm achieved the lowest value of J_1 , demonstrating its effectiveness in minimizing the settling time. This confirms that the proposed methodology ensures convergence of **ga** to the global minimum, avoiding local minima. However, when comparing computational costs, the **ga** required 31.54 minutes, whereas **fmincon** completed the optimization in just 3.04 minutes.

Example 3.4: Continuation

To enhance the system's (3.25) performance in terms of ITAE and ITSE while also reducing energy consumption, the CES and CVS parameters are incorporated into the design. This leads to the following cost function:

$$J_2 = \int_0^t \left(\tau (|e(\tau)| + e(\tau)^2) + u(\tau)^2 + \dot{u}(\tau)^2 \right) d\tau . \quad (3.27)$$

Minimizing the cost function J_2 leads to the convergence of the optimization algorithms toward similar optimal control gains, resulting in comparable closed-loop responses (see Figure 3.12).

A detailed analysis of the generated controllers shows that the controller optimized using **ga**, with parameters $(k_p^*, k_i^*) = (-0.4351, -0.2969)$, achieves the following performance metrics: ITAE = 33.75, ITSE = 14.64, CES = 118.45, and CVS = 1.01. Similarly, using the **fmincon** method with controller parameters $(k_p^*, k_i^*) = (-0.4350, -0.2964)$, the system exhibits an ITAE of 33.77, ITSE of 14.67, CES of 118.41, and CVS of 1.01.

Based on the information presented in Figure 3.12 and above, it is evident that the controller derived through J_2 optimization is almost the same, regardless of the function used (**ga** or **fmincon**). This is evidenced by the reductions in response error and control effort.

To assess the convergence and repeatability of the optimization results, three cases were evaluated for minimizing the cost function J_2 , each involving $N = 16$ independent runs. As expected, the **fmincon** method, being analytical and deterministic, consistently returned the same optimal value when initialized with the same conditions, resulting in a standard deviation of zero. In contrast, the **ga** method was tested under two configurations: Case (I), with a population size of 200 and 10 maximum generations, and Case (II), with a smaller population of 20 and 20 generations.

The results, summarized in Figure 3.13, show that `fmincon` exhibited no variability, which explains the absence of visible box plots or outliers.

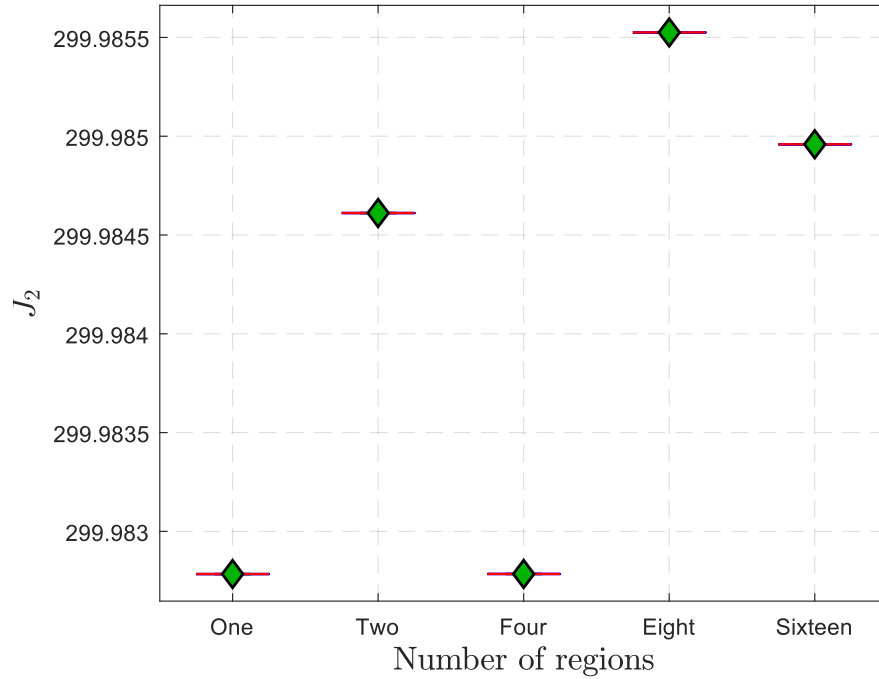
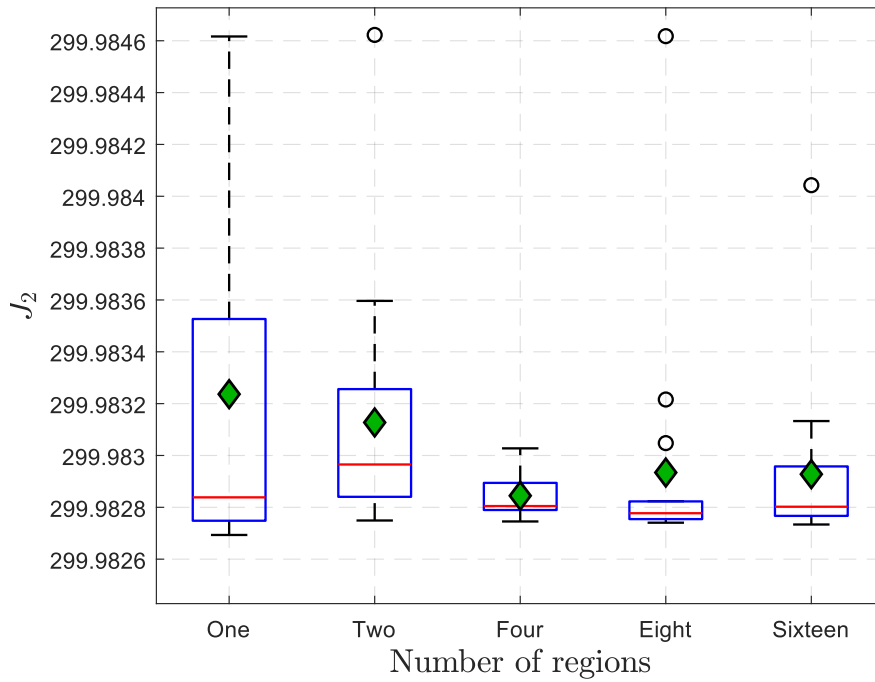
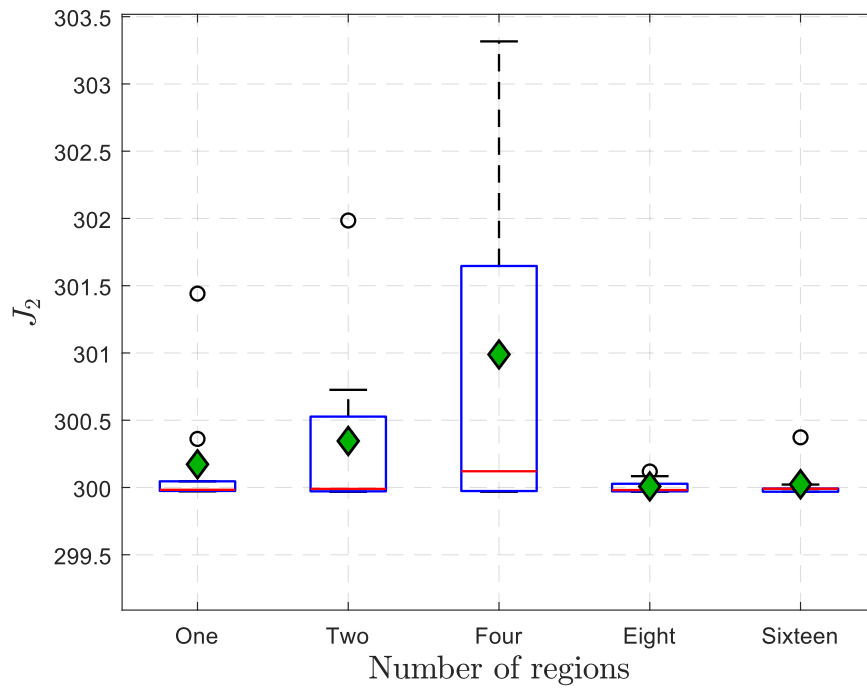


Figure 3.13 / Controller tuning results using `fmincon`.

The `ga` method in Case (I) displayed very low variability (approximately 2×10^{-3}) but required a higher computational cost, around 6.9 minutes per region. Reducing the population size in Case (II) decreased the execution time but introduced greater variability. Increasing the number of subdivisions used to discretize the stability region improved solution accuracy while reducing the overall computational burden. For example, with 16 subregions, the average error in minimizing J_2 using `ga` was approximately 0.0406. However, Case (II) required 655% more computation time than Case (I). These results suggest that discretizing the stability region can lead to more efficient and reliable optimization, especially when computational resources are limited.



(a) Controller tuning results using *ga* (Case I).



(b) Controller tuning results using *ga* (Case II).

Figure 3.15 / Statistical distribution of optimization results across different configurations in tuning methods.

Example 3.5: Sixth Order System

Consider an unstable sixth-order system exhibiting non-minimum-phase behavior, whose open-loop dynamics are represented by the following transfer function

$$G_2(s) = \frac{\frac{13}{10}s^2 - \frac{2}{5}s - 6}{\frac{1}{5}s^6 + \frac{3}{25}s^5 + \frac{293}{200}s^4 + \frac{173}{200}s^3 + \frac{267}{100}s^2 + \frac{53}{40}s + \frac{69}{40}} e^{-\frac{5}{4}s}, \quad (3.28)$$

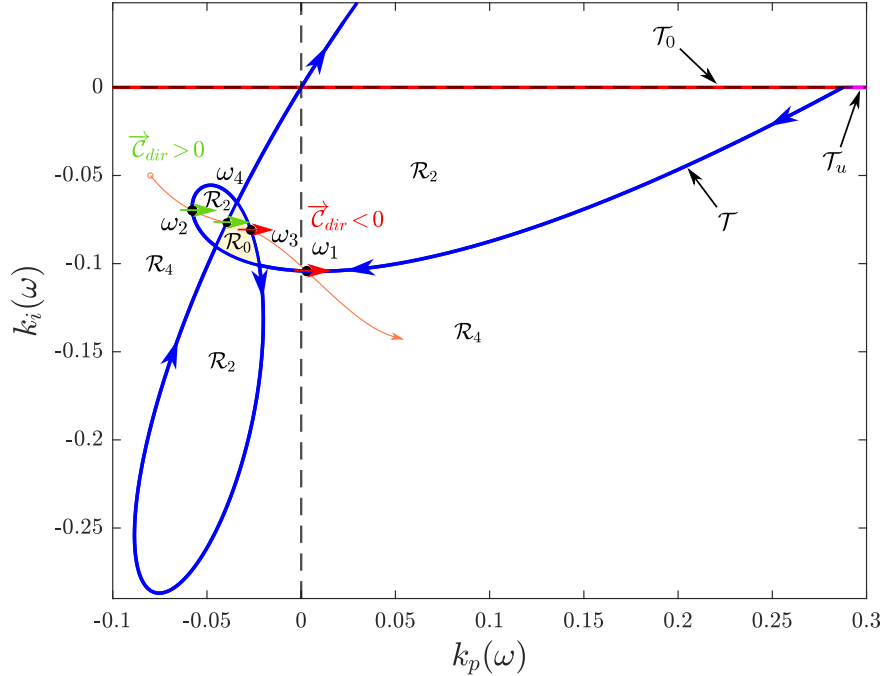
leading to the closed-loop quasi-polynomial

$$\Delta_2(s; k_p, k_i) := sQ_2(s) + P_2(s)(k_p s + k_i)e^{-\frac{5}{4}s}, \quad (3.29)$$

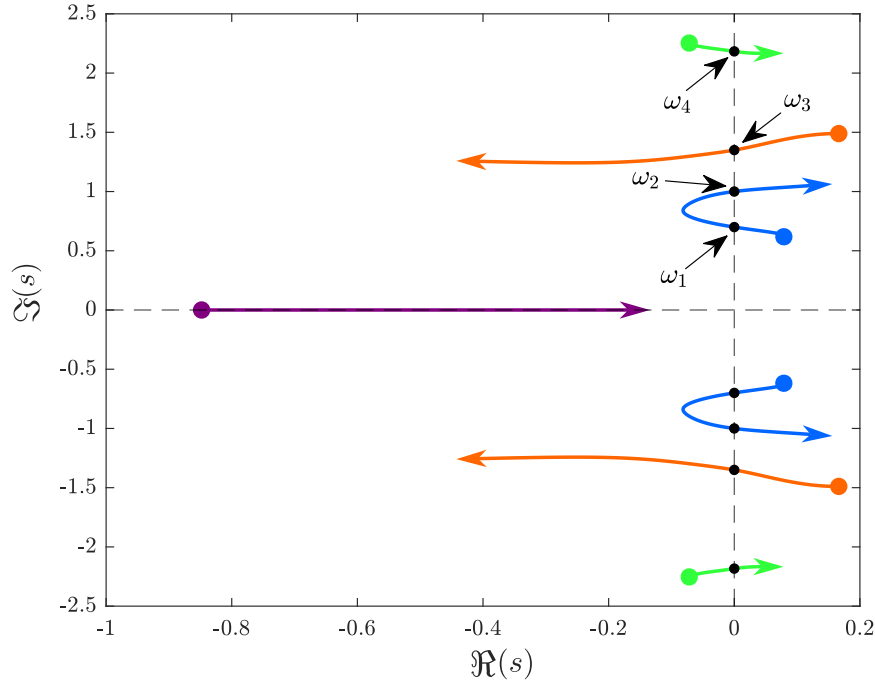
where

$$\begin{aligned} Q_2(s) &:= 40s^6 + 24s^5 + 293s^4 + 173s^3 + 534s^2 + 265s + 345, \\ P_2(s) &:= 260s^2 - 80s - 1200. \end{aligned}$$

By applying Propositions 3.2–3.4, we obtain graphical representations of the parametric plane, as illustrated in Figure 3.16a. Regions that intersect the $\mathcal{T}_u$ curve are identified as unstable and are therefore excluded from the analysis.



(a) Direction of root crossing in the quasi-polynomial (3.29).



(b) Root locus plot of quasi-polynomial (3.29).

Figure 3.17 / Behavior of closed-loop characteristic roots with respect to k_p, k_i in the quasi-polynomial (3.29).

Next, we focus on a subregion where information about the number of roots in the RHP is available. We proceed by increasing k_p , utilizing the result from Proposition 3.5, which determines the direction of root crossing, specifically indicating whether roots are gained or lost during the process.

To demonstrate the effectiveness of Proposition 3.5, we consider the path represented by the orange curve in Figure 3.16a. As $\mathbf{k}$ traverses this path, the system's roots move in the complex plane as illustrated in Figure 3.16b, allowing us to verify the correctness of the crossing directions stated in Proposition 3.5.

As depicted in Figure 3.16a, this path intersects four singular frequencies, $\omega^* = \{0.7, 1, 1.35, 2.183\}$. By applying Proposition 3.5 and Corollary 3.1, following an approach similar to the one used in Example 3.25, we identify the region containing stable gains ($\mathcal{R}_0$).

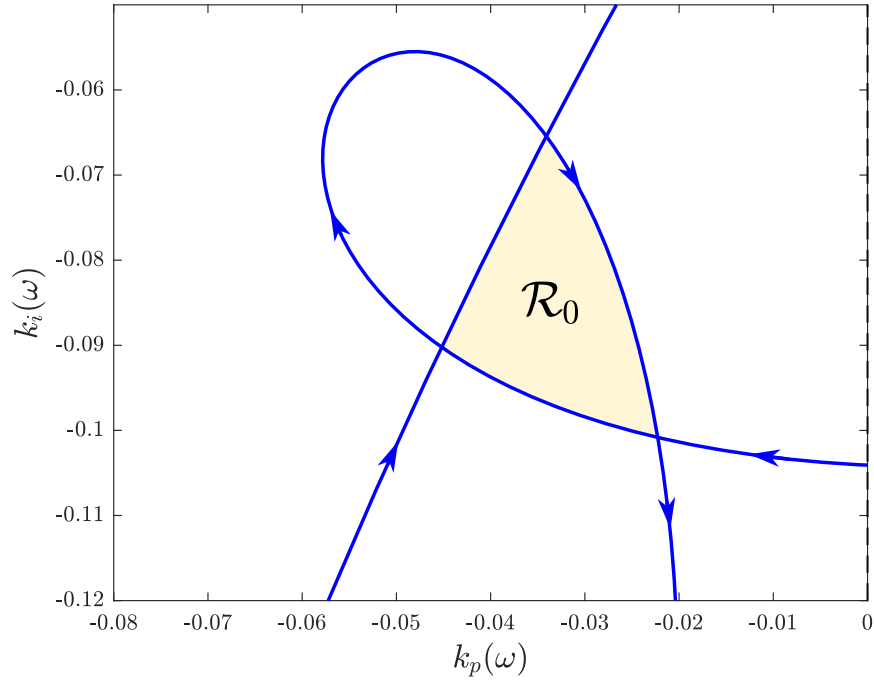


Figure 3.18 / Stability region of quasi-polynomial (3.29).

Additionally, Propositions 3.6 and 3.7 define stable σ -regions for σ values in the range $[0, 0.015]$, as illustrated in Figure 3.19. This significantly reduces the search space, providing essential support for optimization algorithms. For instance, the search can be constrained to a region that guarantees a minimum t_s , thereby facilitating the minimization of other parameters.

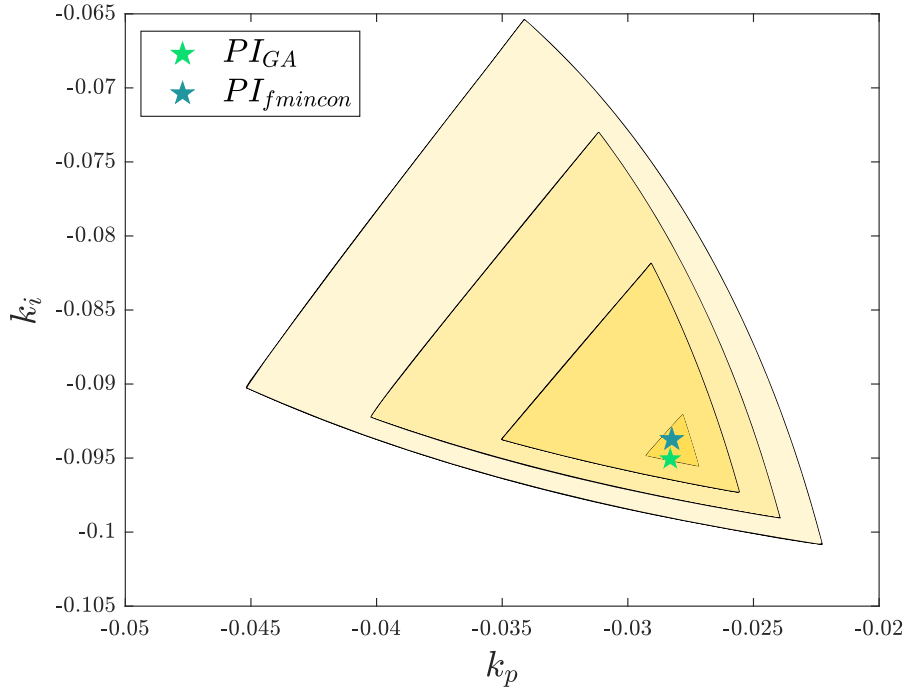
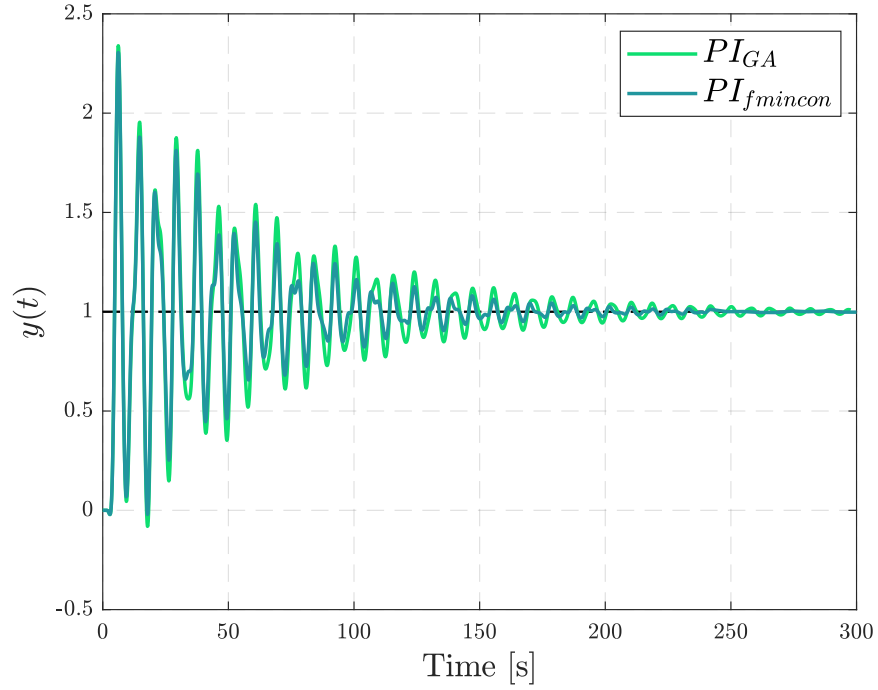


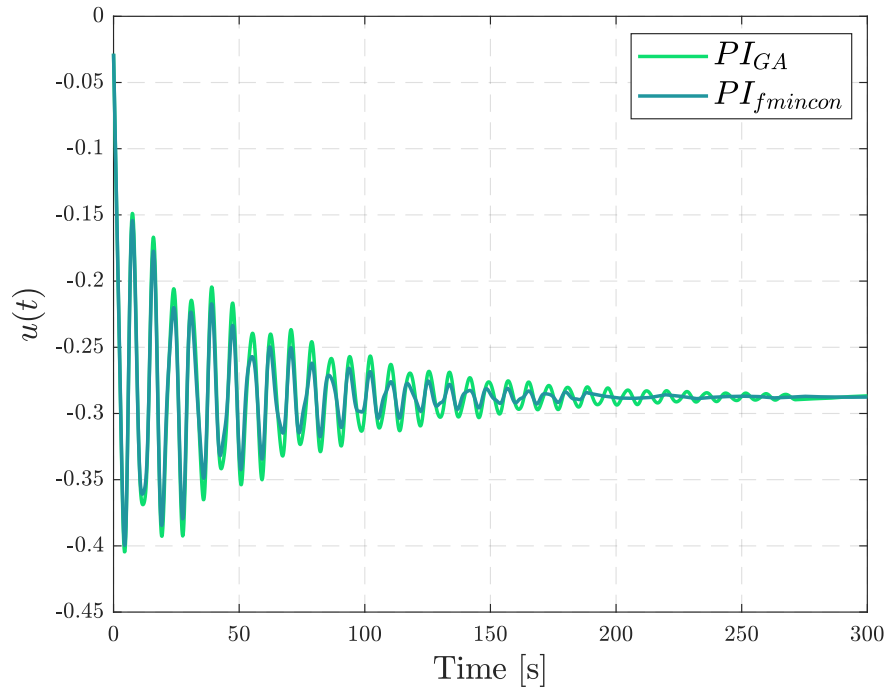
Figure 3.19 / Enlargement of the σ -stability regions.

Figure 3.19 shows the controllers obtained using the **ga** and **fmincon** algorithms with the cost function J_1 , where the region is discretized with $\ell = 20$. The controller optimized by **ga** has gains $(k_p^*, k_i^*) = (-0.02831, -0.0950)$, resulting in a settling time of $t_s = 293.19$ s, while the controller obtained via **fmincon** has gains $(k_p^*, k_i^*) = (-0.02825, -0.0937)$, yielding a settling time of $t_s = 250.08$ s.

As shown in Figure 3.20a, the closed-loop responses of both controllers exhibit a high degree of similarity. However, the controller obtained using the **fmincon** method demonstrates a lower overshoot and a shorter settling time. Notably, both controllers converge to the region with the highest permissible σ value, reinforcing the hypothesis of σ -stability.



(a) Closed-loop response using optimal controllers under σ -stability.



(b) Control signals associated with the responses presented in Figure 3.20a.

Figure 3.21 / Time response of system (3.28) with a step-type entry optimizing J_1 , with $\ell = 20$.

Example 3.6: Haptic device

A haptic interface is a system designed to establish physical interaction between a user and a digital environment through tactile sensations. Its purpose is to reproduce or convey information by means of forces, vibrations, motion, or perceived surface characteristics, thereby enhancing the realism of the interaction. Haptic technologies are used in a broad range of fields, such as virtual and augmented reality, training simulators, gaming, medical applications, and teleoperated systems [12].



Figure 3.22 / Touch™ 3D stylus.

The haptic unit can be represented by a collection of independent linear time-invariant models. Each model describes the mechanical admittance associated with one joint of the device and is expressed as

$$G(s) = \frac{1}{s(ms + b)} e^{-hs}, \quad (3.30)$$

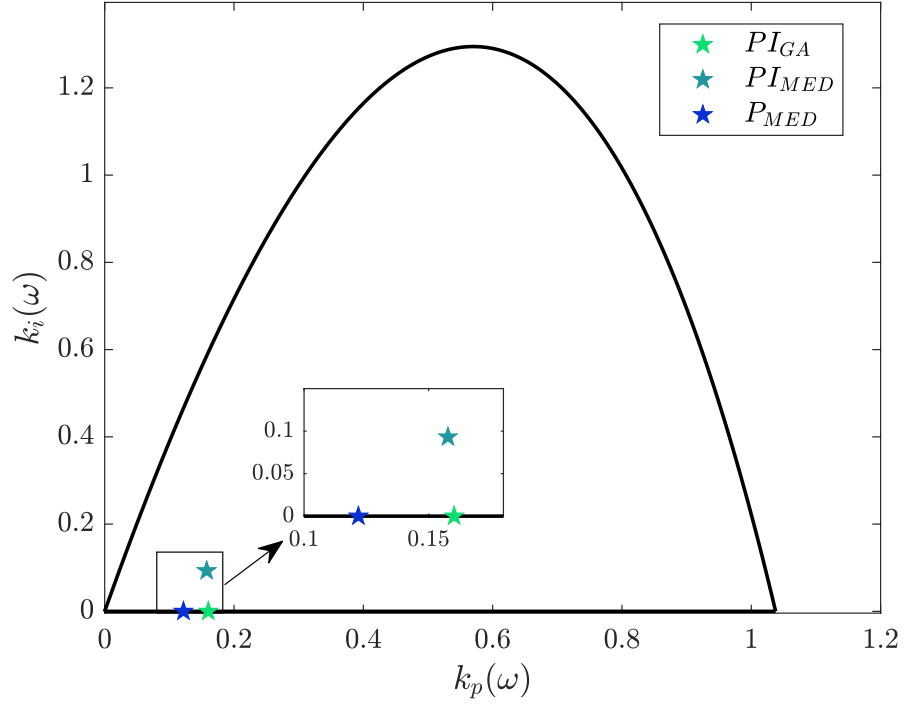
with $m = 0.0131$, $b = 0.0941$, and $h = 0.1$ s.

For this study, the cost function J is evaluated and optimized for several controller configurations to compare the performance achieved by the proposed methodology.

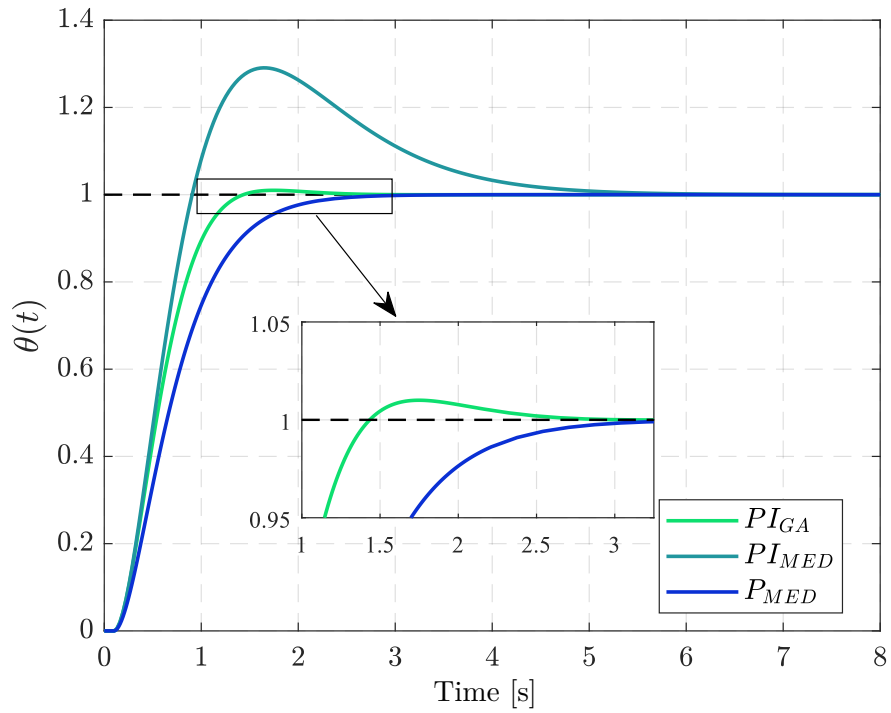
- **t_s vs. MED:** The optimization procedure was carried out under a unit-step reference input. Figure 3.24 presents the resulting closed-loop responses obtained using a PI controller optimized through a genetic algorithm (PI_{GA}). The obtained performance is compared with that of a PI controller (PI_{MED}) and a proportional controller (P_{MED}), both tuned using the Maximum Exponential Decay (MED) criterion.

It can be observed that the PI_{GA} controller achieves a significantly faster convergence toward the reference signal within the 1% settling band. Moreover, this response is obtained with an almost negligible overshoot, highlighting the controller's ability to preserve closed-loop stability while improving

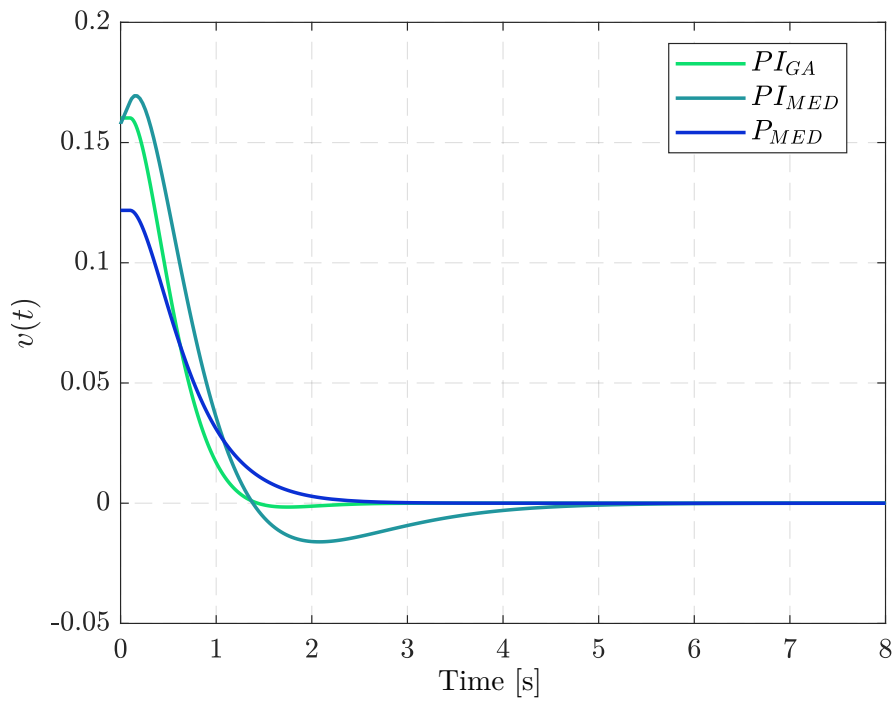
transient performance. In addition, the PI_{GA} controller effectively eliminates the steady-state error, resulting in accurate and robust tracking performance.



(a) Stability region and controllers.



(b) Time responses.



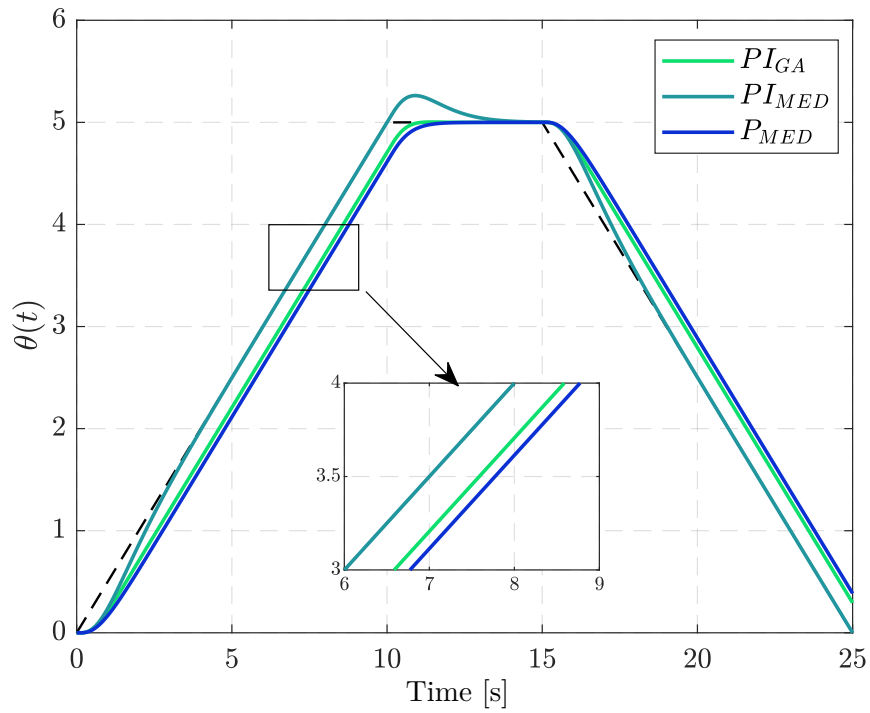
(c) Control efforts.

Figure 3.24 / Total time response obtained for the step-input condition corresponding to Case No. 1.

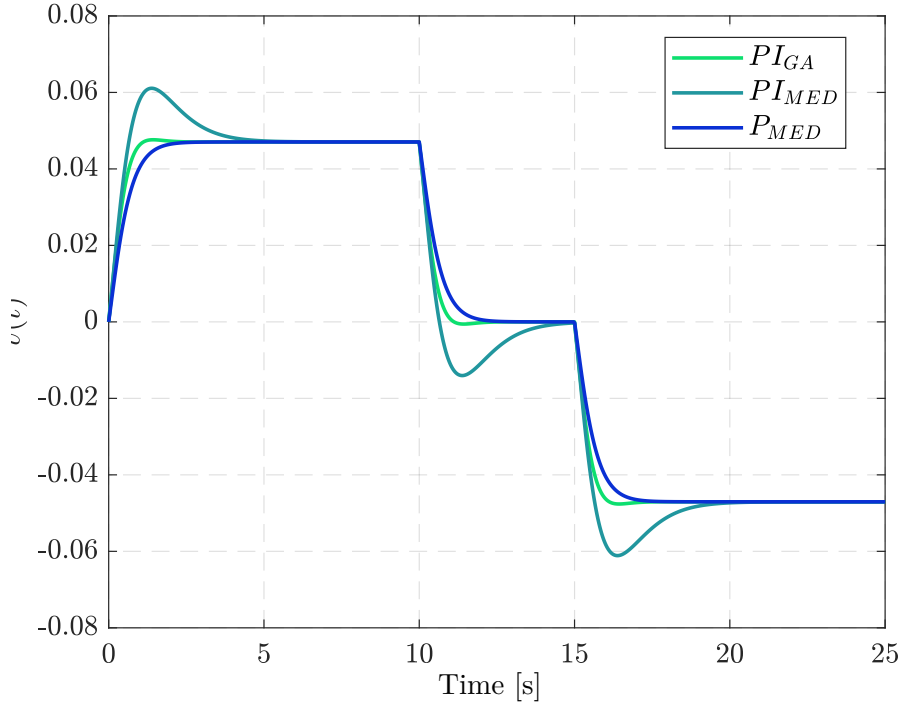
When the output is required to follow a ramp reference while preserving the same controller settings, the resulting closed-loop behavior exhibits a clear variation in its performance, as illustrated in Figure 3.26.

This behavior can be explained by observing that the optimized controller PI_{GA} behaves essentially as a proportional controller, since the corresponding integral gain k_i is approximately zero. Consequently, the controller is unable to guarantee zero steady-state error for ramp-type reference signals.

It is important to emphasize that the synergistic optimization procedure was formulated assuming a unit-step reference input. Therefore, the resulting controller performance strongly depends on the assumptions adopted during the optimization stage. In contrast, the MED-based controller design is derived directly from the spectral properties of the closed-loop system and does not rely on a particular reference signal assumption.



(a) Time responses.



(b) Control efforts.

Figure 3.26 / Total time response with ramp-type entry case No. 1.

- $t_s + \text{ACES}$: The following scenario considers the optimization of the cost function defined in (3.31), where J is a cost function designed to account for the settling time (t_s), the Absolute Control Effort Square (ACES), and the Absolute Control Variation Square (ACVS), defined as

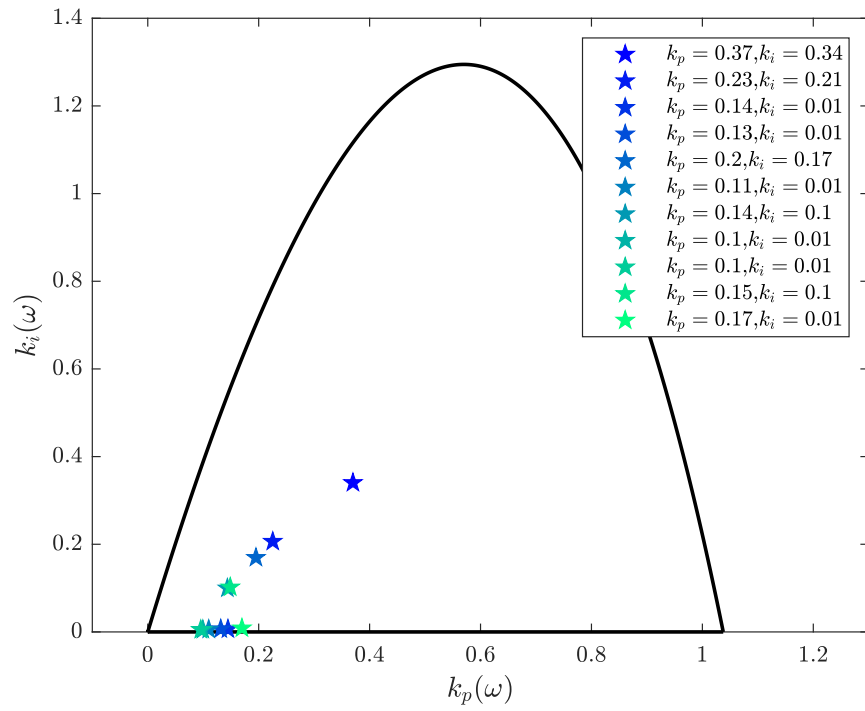
$$J_4 = t_s + \int_0^{t_s} (\beta |u(\tau)|^2 + \gamma |\dot{u}(\tau)|^2) d\tau, \quad \beta, \gamma \in \mathbb{R}_+, \quad (3.31)$$

where

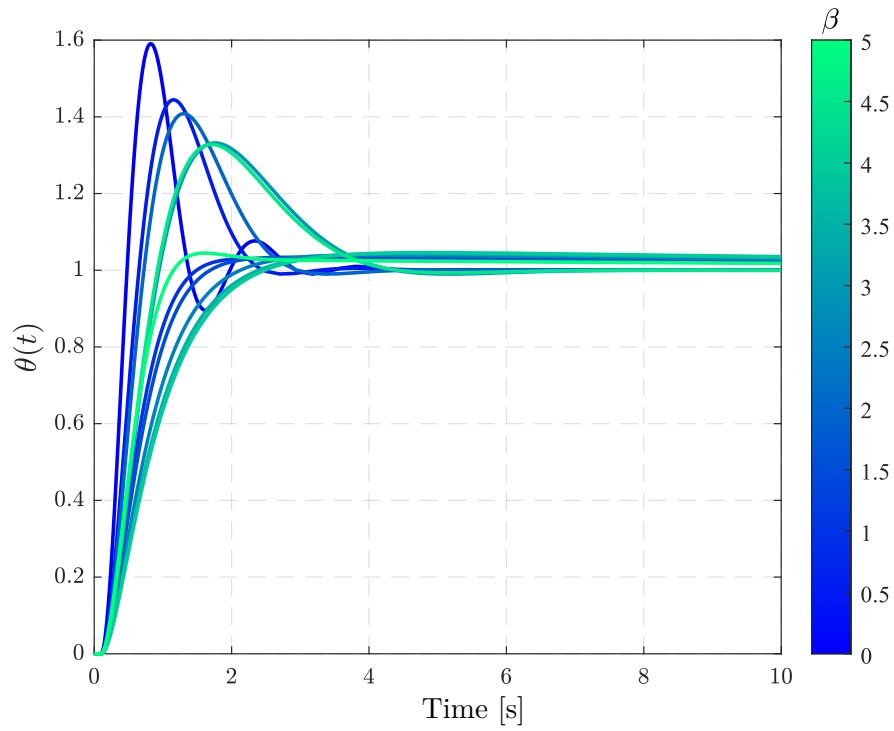
$$t_s := \inf\{\tau \in \mathbb{R}_+ : |y(t) - r(t)| \leq \alpha, \forall t \geq \tau, 0 < \alpha < 1\}.$$

In this case, the optimization problem is solved for $\beta \in [0, 5]$ while fixing $\gamma = 0$. The obtained results are illustrated in Fig. 3.28.

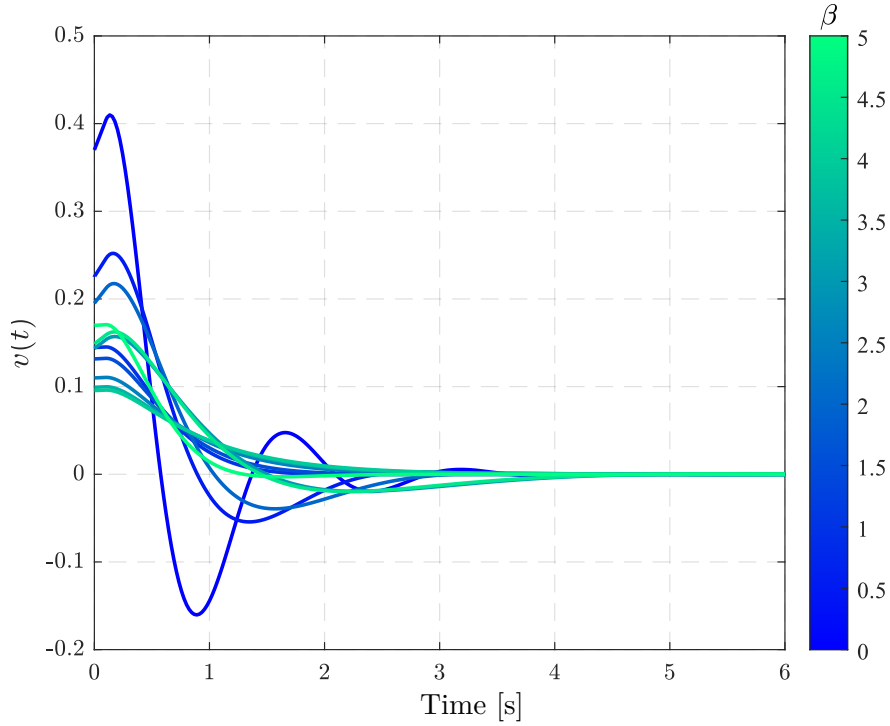
As expected, increasing the value of β produces smoother control signals while simultaneously improving the speed of the transient response. This behavior reflects the trade-off introduced by the ACES criterion between control effort regularization and closed-loop performance.



(a) Stability region and controllers.



(b) Time responses.

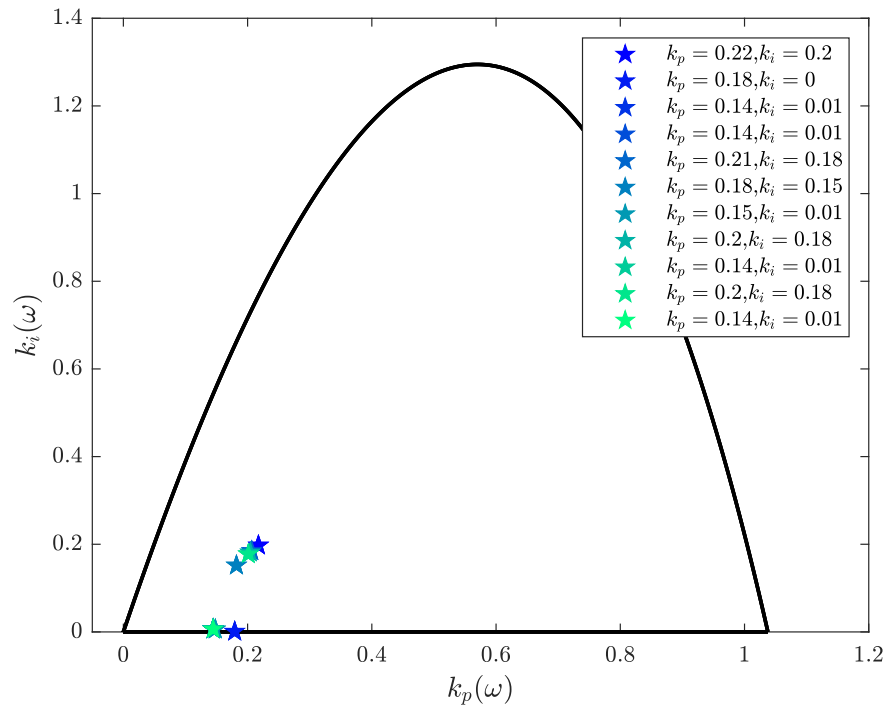


(c) Control efforts.

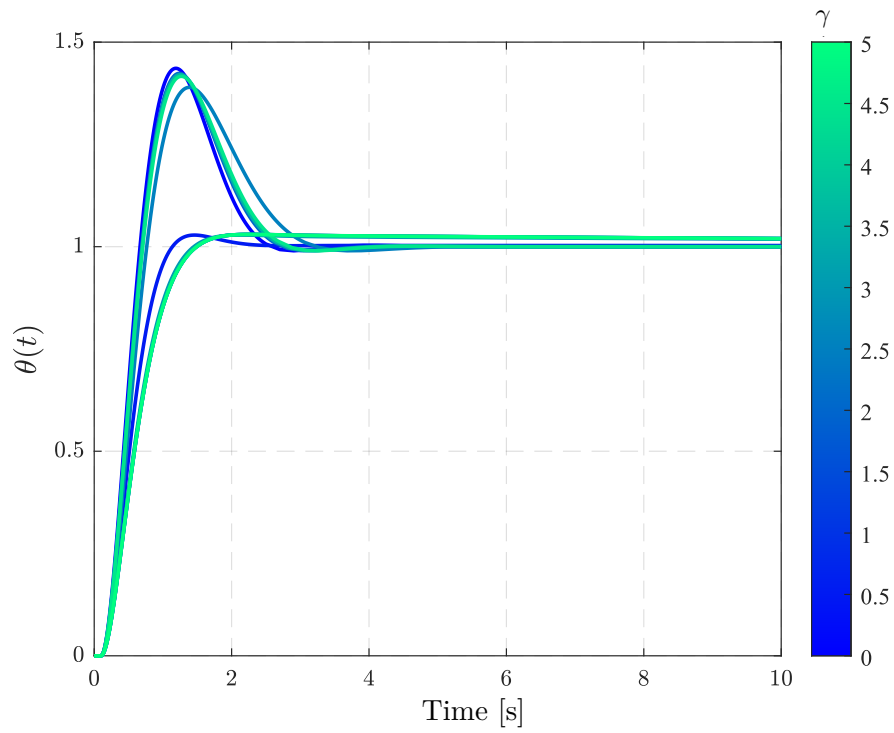
Figure 3.28 / Total time response with step-type entry case No. 2.

- $t_s + \text{ACES} + \text{ACVS}$: To conclude this application, the ACVS criterion is incorporated into the cost function (3.31) by considering $\gamma > 0$. In particular, the optimization is performed using $\beta = 1$ and $\gamma \in [0, 5]$.

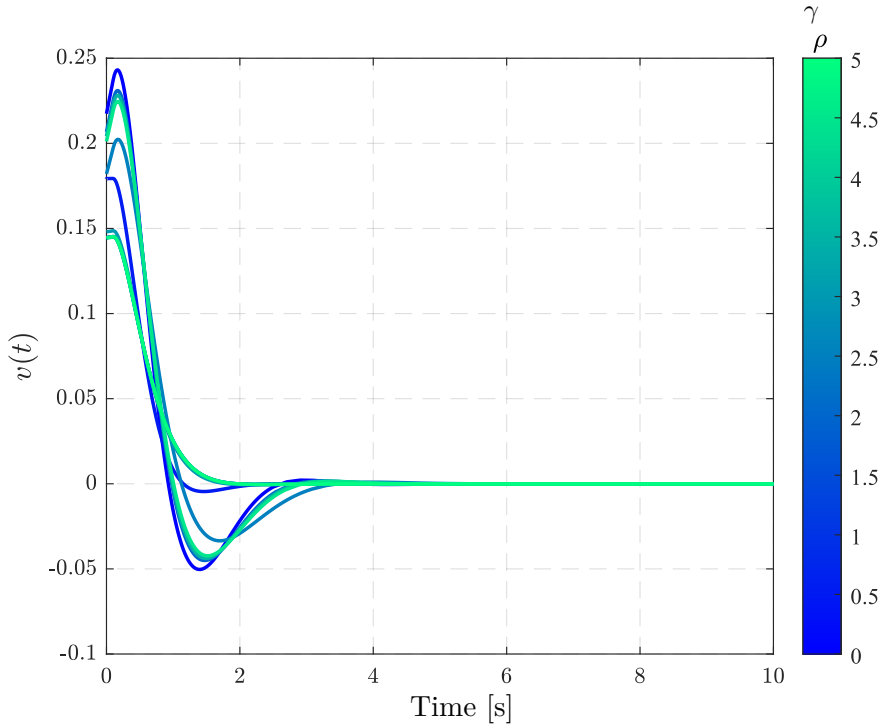
The obtained results are presented in Figure 3.30. As expected, increasing the value of γ produces smoother output responses $y(t)$ and control signals $u(t)$, significantly reducing oscillations and overshoot. This behavior is a direct consequence of penalizing rapid variations in the control effort through the ACVS term.



(a) Stability region and controllers.



(b) Time responses.



(c) Control efforts.

Figure 3.30 / Overall time response obtained for the step-input condition corresponding to Case No. 3.

Example 3.7: Mobile robot

Consider a mobile robot whose kinematic structure follows either a unicycle or a differential-drive configuration, as illustrated in Figure 3.31). To represent its velocity dynamics, the motor associated with each wheel is described by a first-order model including an input delay [68]. The overall robot velocity is then obtained through the corresponding odometric relations.

The transfer function adopted for each DC motor is given by

$$G(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{Ts + 1} e^{-hs}, \quad (3.32)$$

in the reference [68] the values $K = 1.0646$, $T = 0.056$ and $h = 0.084$ were obtained.

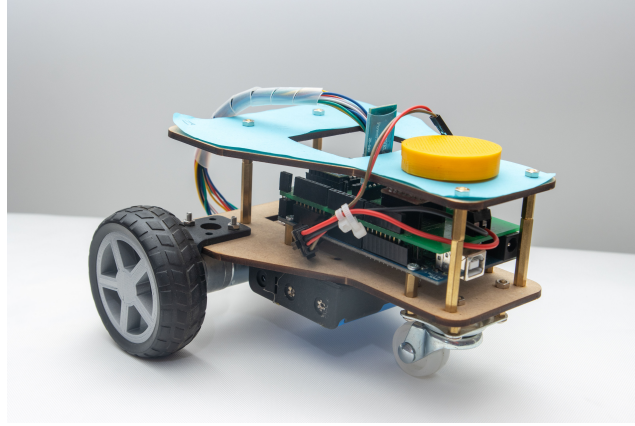
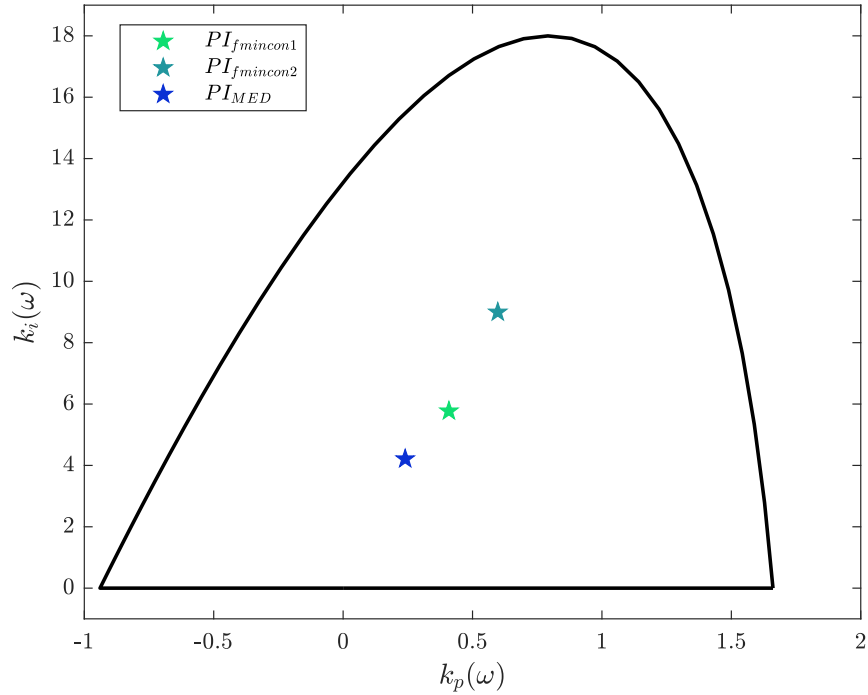


Figure 3.31 / Differential mobile robot prototype.

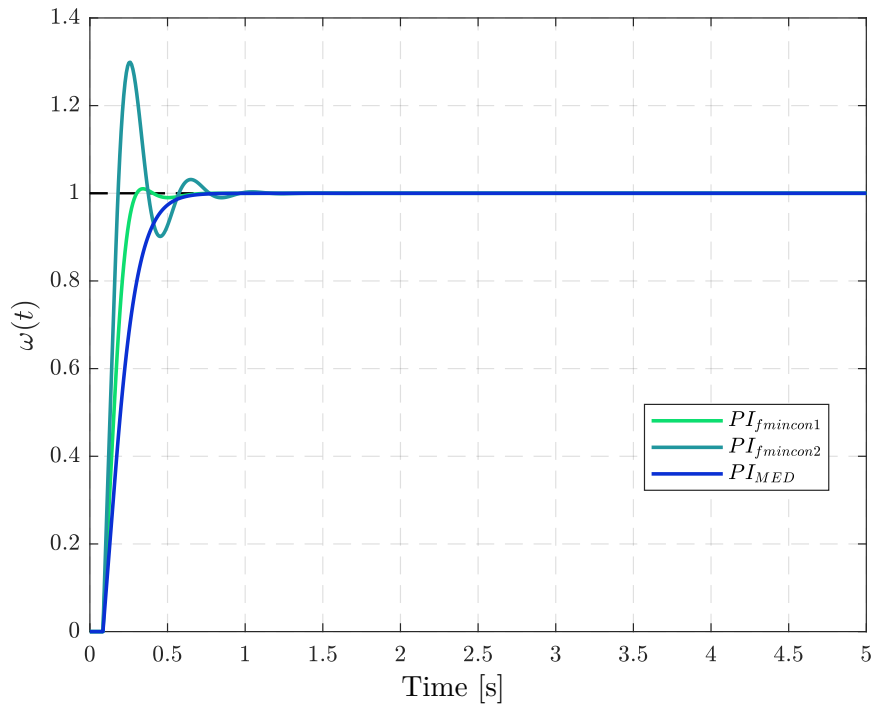
In this approach, four performance parameters are considered: t_s and ISE. Therefore, the cost function is defined as

$$J_5 = t_s(\alpha) + \int_0^{t_s} e(\tau)^2 d\tau . \quad (3.33)$$

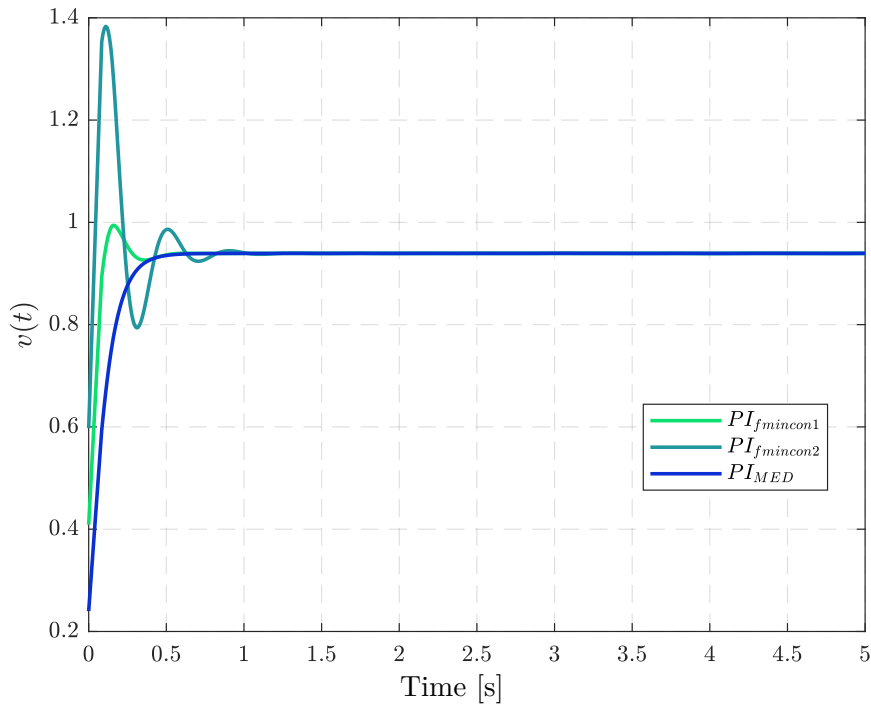
For this application, a trajectory-tracking task has to be fulfilled. One of the most important metrics in this case study is the settling time since the goal is for the robot to follow the reference signal as quickly as possible.



(a) Stability region and controllers.



(b) Time responses.



(c) Control efforts.

Figure 3.33 / Total time response with step-type entry of mobile robot.

The Figure 3.33 shows two controllers obtained via the non-linear optimization algorithm using the *sqp* method; sub-index one was obtained by discretizing the stability region into sub-regions $\mathcal{G}_\ell$ to find different numerical values of the cost function to select the minimum. Sub-index two was obtained using the full search region $\mathcal{G}$ and PI_{MED} it is a tuned controller to obtain the fastest response.

The results show that the controller optimized by the algorithm using $\mathcal{G}_\ell$ search subregions responds the fastest, but since the objective in this scenario is to track the reference with minimum overshoot, the controller selected will be the one obtained by the analytical procedure, PI_{MED} , as previously mentioned.

3.5 Experimental Results

The Qube-Servo 2 integrates a DC motor mechanically connected to an inertial disc, together with a position sensor used to measure the angular displacement. The corresponding transfer function relating the applied voltage to the angular velocity can be expressed as

$$G_1(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{Ts + 1} e^{-\tau s}, \quad (3.34)$$

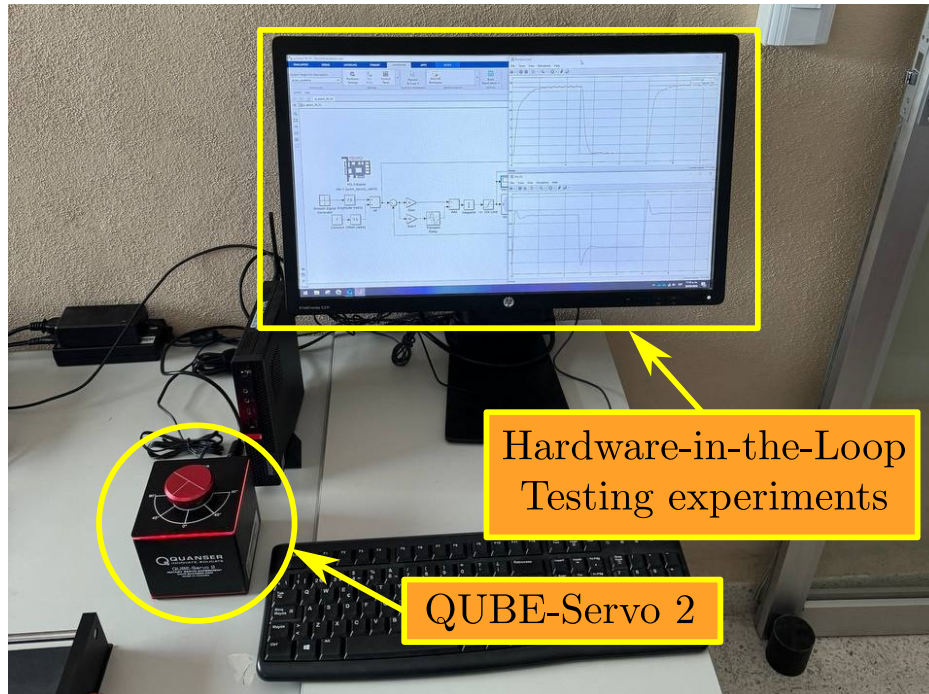


Figure 3.34 / Experimental setup used for validating the controller design.

The platform allows for the practical implementation and assessment of the controller's performance in a controlled environment. All experiments were

executed within MATLAB/Simulink using the ode1 (Euler) solver with a $\Delta t = 2$ ms sampling time.

The system parameters must be accurately identified prior to the controller design. This can be achieved through classical or state-variable approaches, with the controller being either continuous or discrete in nature. In this instance, MATLAB's System Identification Toolbox was employed to derive a first-order delayed model from experimental data gathered within a physical test setup.

$$G_2(s) = \frac{179.93}{s + 7.6} e^{-0.0051s}. \quad (3.35)$$

Fig. 3.35 contrasts the experimental and simulated responses to a dynamic input over time. The acquired results yield a correlation factor of $R^2 = 0.9875$.

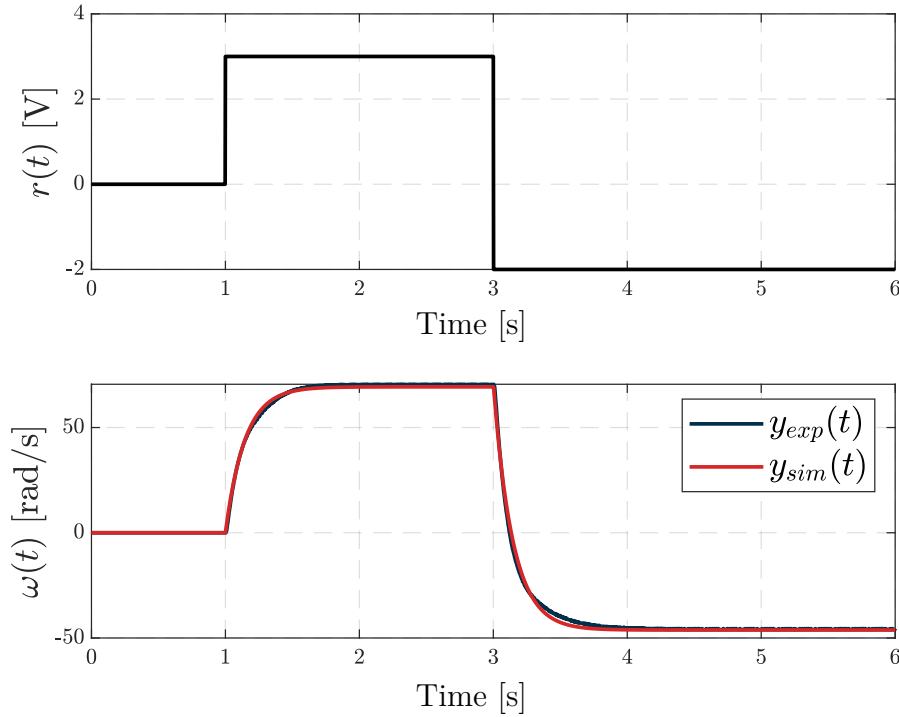


Figure 3.35 / Process parameter identification using a real-time hardware in the loop with a simple simulation testing.

Experiment 3.1: Position Control

The voltage-to-velocity transfer function of the system is given by (3.35). If you now want to obtain the position, you only need to integrate as shown below.

$$G_3(s) = \frac{\Theta(s)}{V(s)} = \frac{1}{s} \frac{\Omega(s)}{V(s)} = \frac{23.651}{s(0.1314s + 1)}. \quad (3.36)$$

In this application, the performance indices considered are the settling time t_s with $\alpha = 0.01$, as well as CES and CVS. We explore various combinations of

weight parameters, with $\beta \in [0, 3]$ and $\rho \in [0, 5]$, in the cost function:

$$J_3 = t_s(\alpha) + \int_0^t (\beta u(\tau)^2 + \gamma \dot{u}(\tau)^2) d\tau. \quad (3.37)$$

Using the transfer function (3.36) and Algorithm 1, we discretize the stability region with $\ell = 1$. After optimizing all parameters, 24 experiments were conducted, and their results are presented in Figure 3.36, providing a comprehensive analysis of the control system's behavior under various combinations of β and γ .

The controller's performance is evaluated based on five parameters, as detailed in Table 3.1, along with three additional parameters:

- $OS = \frac{y(t_p) - y(\infty)}{y(\infty)} \cdot 100\%$
- $IAE = \int_0^t |e(\tau)| d\tau$
- $ISE = \int_0^t e(\tau)^2 d\tau$

The normalized values of these parameters are illustrated in Figure 3.36.

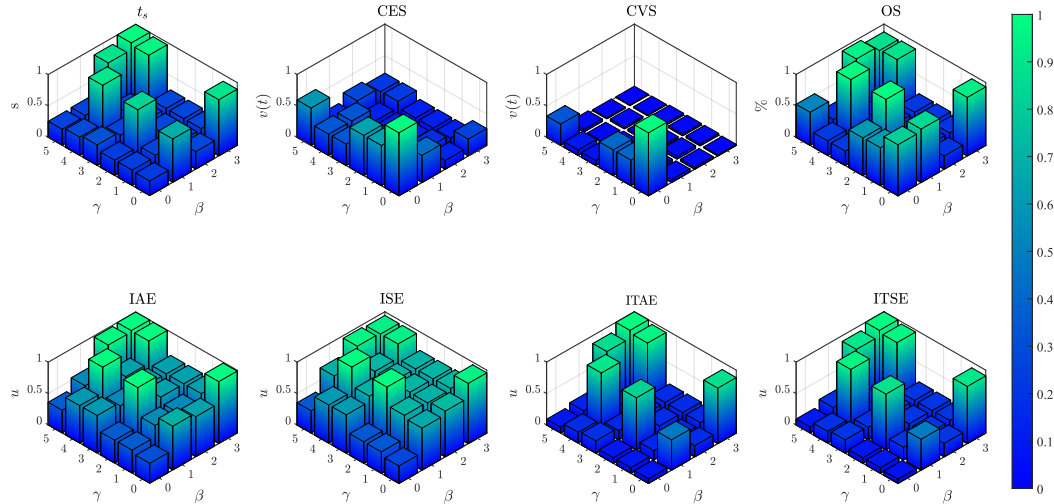
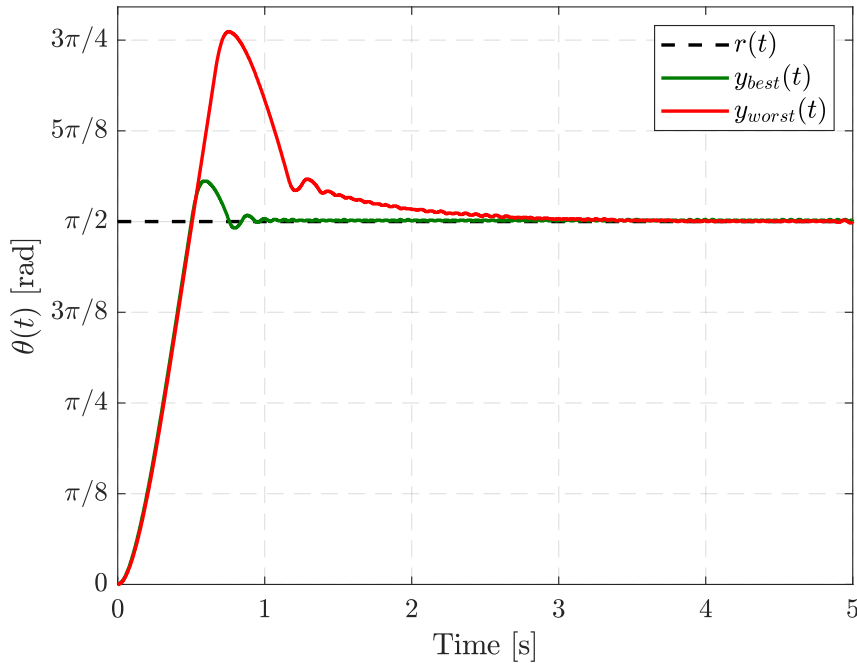


Figure 3.36 / Performance parameter analysis for the control system (3.36) across 24 experiments.

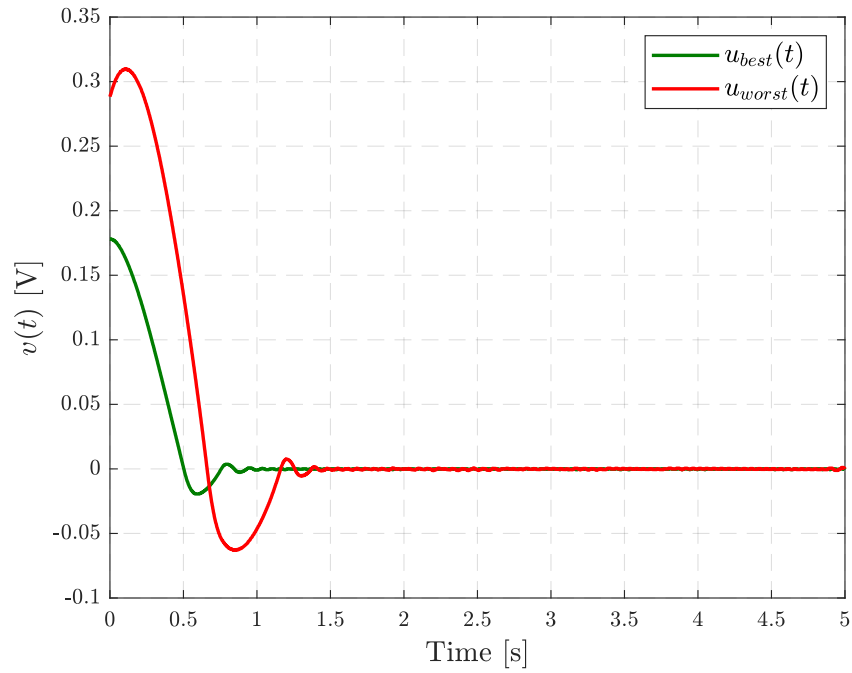
After evaluating the performance measures reported in Table 3.1 together with the three supplementary parameters, the L_2 norm of the resulting eight indicators is employed to rank the 24 experimental trials and determine the best- and worst-performing PI controller configurations. The L_2 norm is defined as:

$$\mathcal{I} = \sqrt{t_s(\alpha)^2 + CES^2 + CVS^2 + IAE^2 + ISE^2 + ITAE^2 + ITSE^2 + OS^2}. \quad (3.38)$$

Therefore, among the evaluated controllers, the best PI controller is the one that minimizes (3.38). This minimum value is achieved when $(\beta, \rho) = (1, 5)$, corresponding to the optimal tuning parameters $(k_p^*, k_i^*) = (0.1134, 0.001)$. Conversely, the worst PI controller is the one that maximizes (3.38), which occurs for $(\beta, \rho) = (1, 2)$ with the tuning configuration $(k_p, k_i) = (0.1835, 0.2716)$. Figure 3.38 illustrates the time response obtained when implementing both the best and worst PI controllers in the experiments.



(a) Time-domain responses of the control system.



(b) Time-domain responses of the control effort for the evaluated controllers.

Figure 3.38 / Best and worst response of Qube-Servo 2.

The presented results stem from the experiments that can be observed in https://youtu.be/10cA_SMvHWY.

Analysis and Design of Delay-Based Controllers for Delay-Free Systems

Mathematics is the language with which God has written the universe.

Galileo Galilei

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This chapter investigates the use of intentional delays as additional tuning parameters for the control of delay-free systems. The proposed delay-based controllers are designed to improve transient performance, robustness, and exponential convergence rates without relying on derivative actions. Stability regions are characterized analytically using frequency-domain techniques and stability crossing boundaries. Furthermore, optimization-based methodologies are employed to determine controller parameters that maximize performance indices such as spectral abscissa or settling time. The developed results demonstrate that intentionally introduced delays can significantly enhance closed-loop behavior in a wide class of dynamical systems.

Consider the P δ I control structure introduced in [69]. The associated control law can be written as

$$u(t) = k_p e(t) + k_i \int_0^t e(v - h) dv, \quad h > 0. \quad (4.1)$$

The transfer function of P δ I controller is

$$C(s) = k_p + \frac{k_i}{s} e^{-hs}, \quad k_p, k_i \in \mathbb{R}, h \in \mathbb{R}_+. \quad (4.2)$$

4.1 Integral Second-Order System

In the design of controllers for linear systems, it is well established that the settling time (t_s) depends on the location of the closed-loop poles of the characteristic equation. In general, placing the dominant poles further to the left in the complex plane leads to a smaller t_s . To reduce the settling time, the controller gains are adjusted to achieve what is referred to as the “*maximum exponential decay*”, denoted by σ_m , which corresponds to the real part of the dominant poles. This value indicates how far the dominant poles can be shifted to the left, resulting in the fastest system response.

Before presenting the main results, an example is provided to show that σ_m can reach higher values when using a P δ I controller compared to a classical PI controller.

Problem formulation

Consider the family of second-order linear time-invariant systems represented by the transfer function

$$G(s) = \frac{K}{s(Ts + 1)}, \quad (4.3)$$

where $K > 0$ and $T \in \mathbb{R}_+$.

Problem 4.1: Stability conditions

The objective is to derive the analytical conditions on the parameters (k_p, k_i, h^*) such that the P δ I controller stabilizes the closed-loop system shown in Figure 4.1, for a fixed delay value h^* .

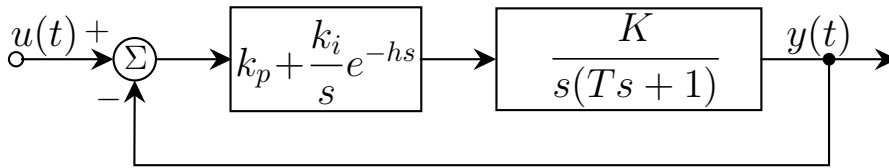


Figure 4.1 / General structure of a standard closed-loop control configuration.

Problem 4.2: Optimize exponential decay

By considering the closed-loop system shown in Figure 4.1, the objective is to determine the corresponding maximum exponential decay, denoted by σ_m .

Based on the problems outlined above, a motivating example is presented to demonstrate the feasibility of the proposed approach.

Example 4.1: Integral Second Order System

Consider the following second-order system

$$G(s) = \frac{200}{s^2 + 10s}, \quad (4.4)$$

together with a classical PI controller given by

$$C_{PI}(s) = k_p + \frac{k_i}{s}, \quad (4.5)$$

where $k_p, k_i \in \mathbb{R}$. Assuming unitary negative feedback, and following the procedure described in [8] to compute the maximum exponential decay σ_m , a value of $\sigma_{m_{PI}} = \frac{10}{3}$ is obtained (the subscript PI indicates the classical PI controller).

Figure 4.2 shows the closed-loop responses of system (4.4) for different choices of the controller (4.5). The results indicate that the fastest response is achieved when σ reaches its maximum value $\sigma_{m_{PI}}$. This corresponds to placing the dominant poles of system (4.4) as far as possible to the left in the complex plane.

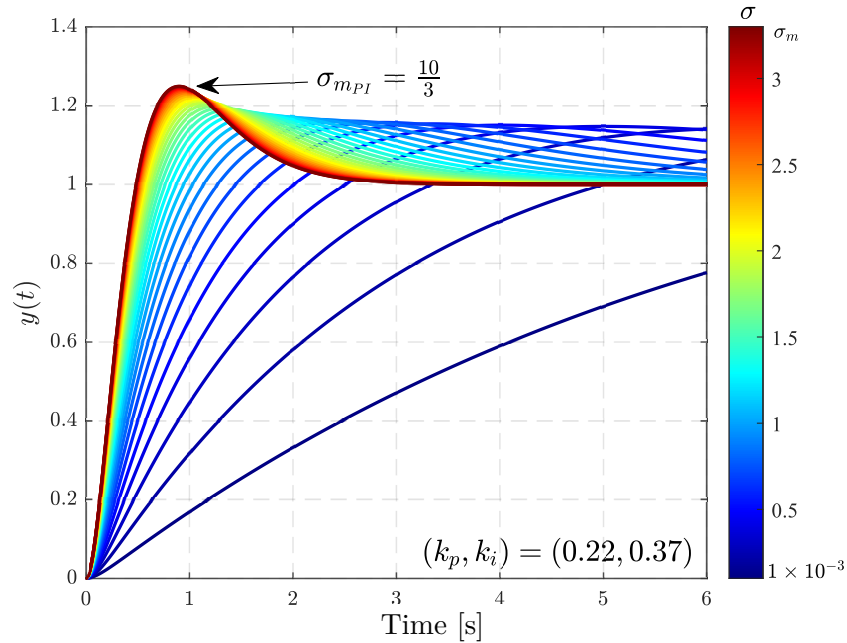


Figure 4.2 / Time step response closed-loop system of (4.4) for different controllers (4.5) by increasing the σ value.

Now, consider an arbitrary stabilizing PI controller and introduce a delay in its integral term as (4.2). To analyze the effect of the time delay on the control performance, Figure 4.3 shows the closed-loop response for different values of

h in (4.2). It can be observed that, as the delay increases, the overshoot is reduced and the response improves up to a certain point. However, beyond a specific value of h , oscillations begin to appear and grow, until the system reaches a critical delay at which it becomes unstable.

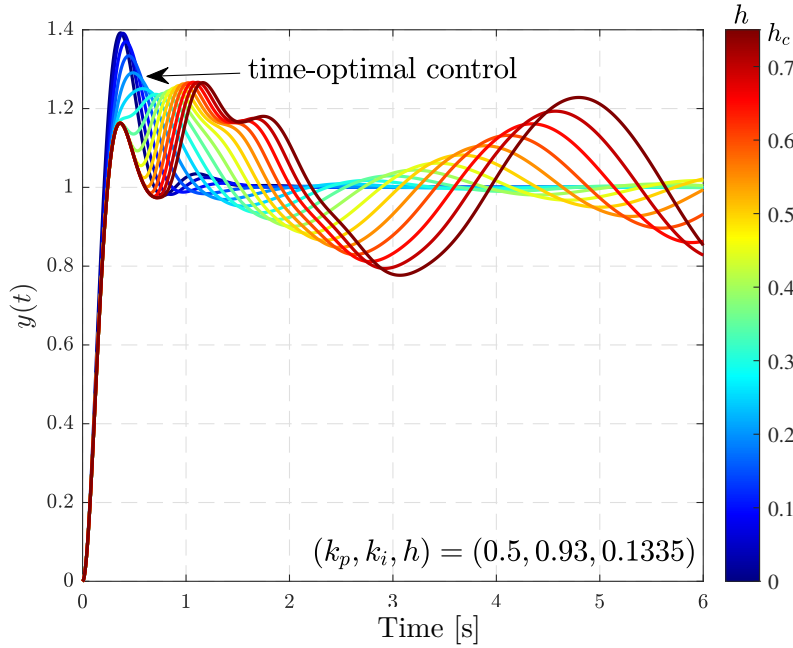


Figure 4.3 / Time-domain response of the closed-loop system obtained by increasing the delay h while keeping the stabilizing gains (k_p, k_i) fixed.

As h increases, the response reaches an optimal behaviour then it becomes unstable.

The response characteristics of the fastest PI controller are compared with those obtained using the time-optimal P δ I controller, which provides the minimum settling time t_s , as illustrated in Figure 4.3.

In particular, while the fastest regulation obtained with the PI controller in Figure 1 yields a spectral abscissa of $\sigma_m = \frac{10}{3}$, the time-optimal response shown in Figure 4.3 achieves a spectral abscissa of $\sigma = 4.14$. These results indicate that the P δ I controller can provide improved performance and faster regulation compared to the fastest PI controller.

However, as illustrated in Figure 4.3, the value of the delay h must be carefully selected, since it may also lead to instability [8] [35]. Therefore, it is necessary to determine an appropriate value of h for fixed controller gains.

If a root locus analysis is performed, the behavior of the system poles can be observed as shown in Figure 4.4.

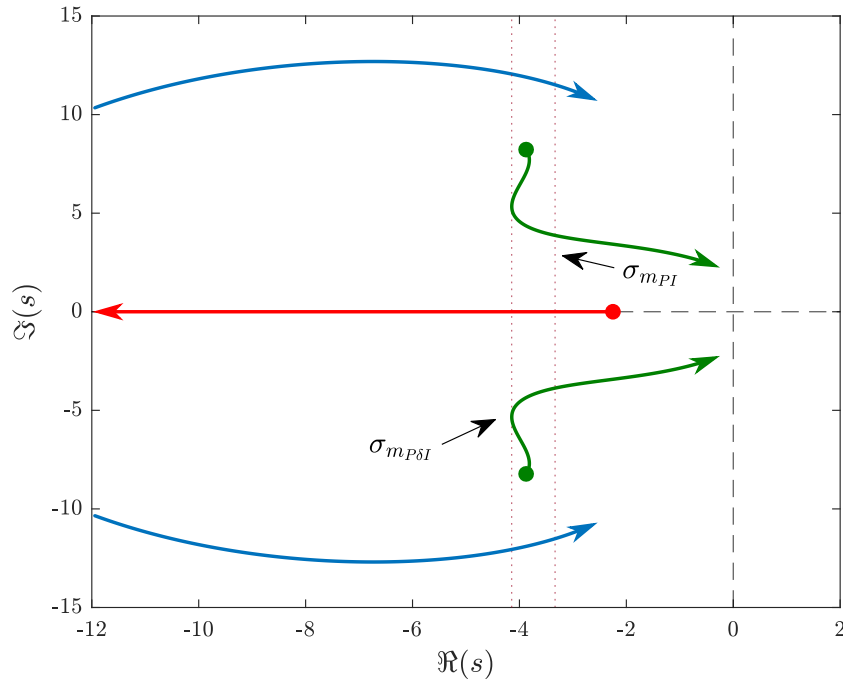


Figure 4.4 / Root movement of the system (4.4) under the controller (4.2) for increasing values of h .

As shown in the Figure 4.4, the maximum σ achieved with a PI controller is lower than the one obtained with a P δ I controller. Therefore, it can be concluded that the exponential decay rate of the response is higher when using the delayed controller.

Therefore, to support the previous conclusion, simulations were carried out under the same initial conditions to compare the response performance of a PI controller and a P δ I controller. The results are shown in Figure 4.5.

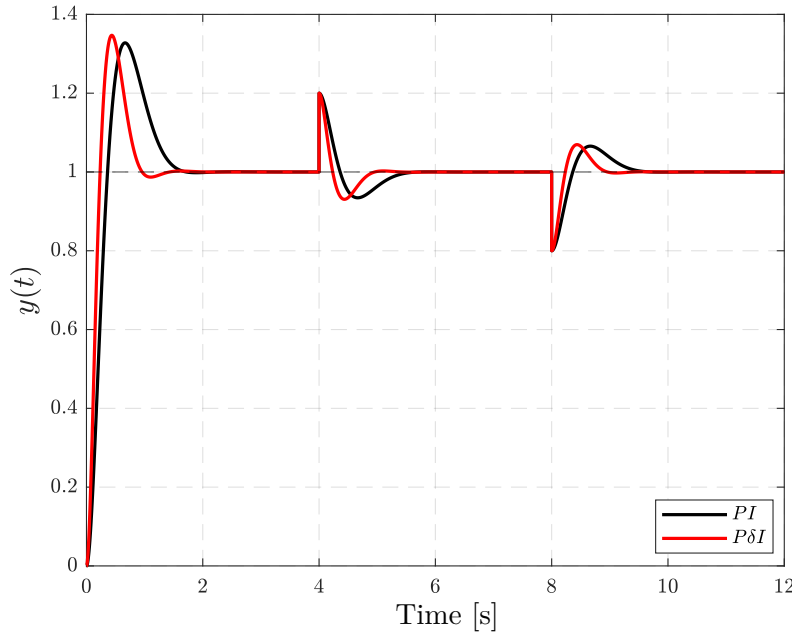


Figure 4.5 / Comparison between a PI controller and a PδI controller under additive output disturbances.

It can be observed that the delayed controller achieves a smaller settling time t_s , indicating a faster response.

Analytical approach

The principal results developed in this section are presented next. The discussion begins with several basic definitions and preliminary results that will subsequently be employed to derive the stability maps in the controller parameter space, following the same general procedure adopted in Chapter 3.

DEFINITION

4.1

The characteristic quasi-polynomial of the closed-loop system (4.3), is given by:

$$\Delta(s; k_p, k_i, h) := Ts^3 + s^2 + Kk_p s + Kk_i e^{-hs}. \quad (4.6)$$

PROPOSITION

4.1

Let Δ be a given quasi-polynomial of retarded-type. Then, Δ is exponentially stable if and only if all its roots have a negative real part, that is, if $s^* \in \mathbb{C}$ is a root of Δ for a fixed $(k_p, k_i, h) \in \mathbb{R}^2 \times \mathbb{R}_+$, then

$$\Delta(s^*; k_p, k_i, h) = 0 \iff \Re(s^*) < 0.$$

DEFINITION

4.2

The *frequency crossing set* $\Omega \subset \mathbb{R}$ is defined as the set of all $\omega \in \mathbb{R}$ for which there exists at least one triplet $(k_p, k_i, h) \in \mathbb{R}^2 \times \mathbb{R}_+$ such that

$$\Delta(i\omega; k_p, k_i, h) = 0. \quad (4.7)$$

Remark 4.1. It should be noted that Definition 4.2 only considers positive frequencies. This is because the following property holds:

$$\Delta(-i\omega; k_p, k_i, h) = \overline{\Delta(i\omega; k_p, k_i, h)}. \quad (4.8)$$

┘

DEFINITION

4.3

For a fixed $h^* \geq 0$, the *stability crossing curves* (also called stability boundaries in this work) $\mathcal{K}$ is the set of all parameters $\mathbf{k} = (k_p, k_i) \in \mathbb{R}^2$ for which there exists at least one $\omega \in \Omega$, such that, $\Delta(i\omega; k_p, k_i, h^*) = 0$. Moreover, any point $\mathbf{k} \in \mathcal{K}$ is known as a *crossing point*.

With all the previous definitions in place, it is now possible to generate stability maps in the controller parameter space, which allow the identification of stabilizing controllers.

To build stability maps in the parameter space, it is important to observe that the roots of a quasipolynomial can cross the imaginary axis in three different ways: at the origin of the complex plane (real root boundary), at infinity (infinite root boundary), and at a finite set of singular frequencies (complex root boundary). These cases are illustrated in Figure 4.6. The relationship between the complex plane and the parameter space is obtained through the following analytical results.

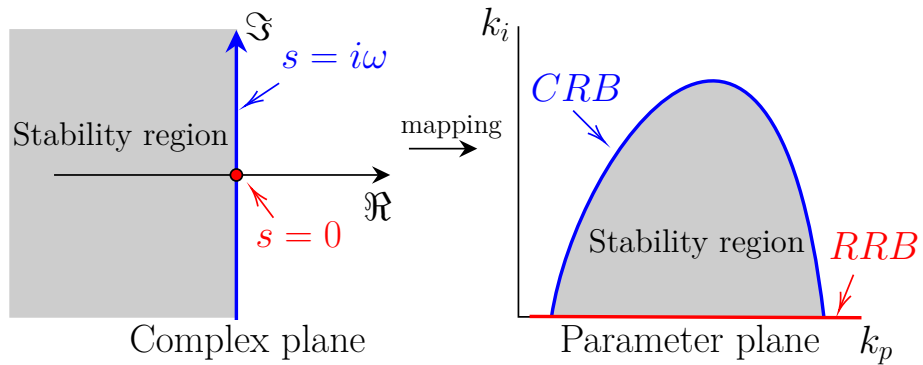


Figure 4.6 / Mapping of the stability crossing boundaries from the complex plane to the parameters plane.

$s = i\omega$ refers to the complex root boundary (CRB) and $s = 0$ to the real root boundary (RRB).

Stability crossing curves

PROPOSITION

4.2

Let $h > 0$ be a fixed value and $\Omega = \bigcup_{\ell} \Omega_{\ell}$ for $\ell \in \mathbb{N}$, where the sets $\Omega_{\ell} \subset \mathbb{R}_+$ are given by

$$\Omega_{\ell} := \left\{ \omega \in \mathbb{R}_+ \mid \omega \in \left(\frac{\pi(\ell-1)}{2h}, \frac{\pi\ell}{2h} \right) \right\}. \quad (4.9)$$

Then, ω is a crossing frequency if and only if $\mathbf{k}(\omega; h) := [k_p(\omega; h) \ k_i(\omega; h)]^{\top}$, where

$$k_p(\omega; h) := \frac{\omega(T\omega + \tan(h\omega))}{K}, \quad (4.10a)$$

$$k_i(\omega; h) := \frac{\omega^2 \sec(h\omega)}{K}, \quad (4.10b)$$

defines a crossing point $\mathbf{k}(\omega, h) \in \mathcal{K}$.

Proof. Consider the characteristic equation (4.6). From Definition 4.3, the set of crossing points $\mathbf{k} \in \mathcal{T}_{\omega}$ consists of all pairs (k_p, k_i) that satisfy equation (4.7) when $s = i\omega$. Therefore, separating the equation $\Delta(i\omega; k_p, k_i) = 0$ into its real and imaginary parts leads to

$$\Re(\Delta(i\omega; k_p, k_i)) = 0, \quad (4.11a)$$

$$\Im(\Delta(i\omega; k_p, k_i)) = 0. \quad (4.11b)$$

Solving (4.11a) and (4.11b) with respect to k_p and k_i , after straightforward algebraic manipulations, results in expressions (4.10a) and (4.10b). $\square$

PROPOSITION

4.3

Let $h \in \mathbb{R}_+$ be a fixed delay value. Then, the line $\mathbf{k}_0 \in \mathbb{R}^2$ defined by

$$\mathbf{k}_0 := \left\{ \begin{bmatrix} k_p \\ 0 \end{bmatrix} \mid k_p \in \mathbb{R} \right\}, \quad (4.12)$$

defines a stability crossing curve. Furthermore, this line characterizes a crossing through the origin of the complex plane.

Proof. The proof follows the same reasoning as in the CRB case, considering the condition $\Delta(0; k_p, k_i, h) = 0$. $\square$

The tuning of the P δ I controller is carried out using a geometric approach based on the so-called $\mathcal{D}$ -partitions [37]. This technique divides the parameter space of a characteristic polynomial or quasipolynomial into regions according to changes in the degree of instability. When these partitions are constructed with respect

to a parameter σ , the resulting regions are referred to as σ -stable regions. These regions are characterized using the spectral abscissa and the concept of σ -stability, which are defined in Chapter 2, specifically in the Definitions 2.19-2.20.

σ -stability

For a fixed value $h \geq 0$, let $\mathcal{K}_\sigma$ denote the set of all gains $\mathbf{k}$ such that the quasipolynomial (4.6) has at least one root located on the vertical line of the complex plane defined by $\mathbf{L}_\sigma := -\sigma + i\omega$, for all $\omega \in \mathbb{R}$. This vertical line is referred to as the σ -axis (see Figure 4.7).

Additionally, define the set $\Omega_\sigma \subseteq \mathbb{R}_+$ as

$$\Omega_\sigma := \{\omega \in \mathbb{R} \mid \sigma \tan(h\omega) \neq \omega\}. \quad (4.13)$$

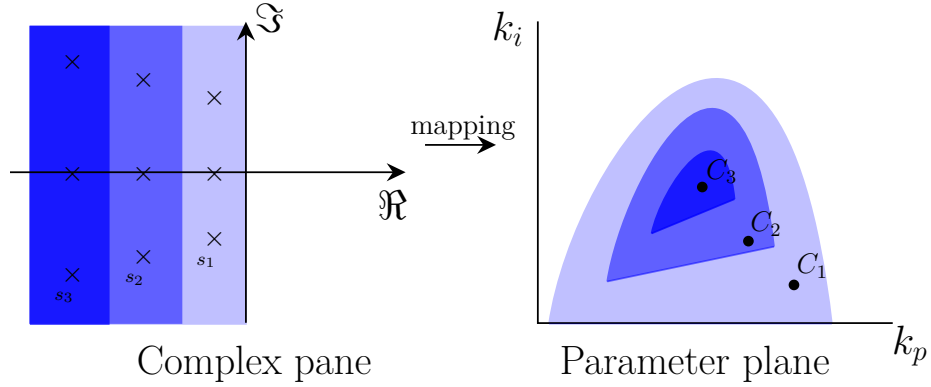


Figure 4.7 / Stability regions with different σ -stability degrees.

The figure shows three sigma-stable zones s_1, s_2, s_3 in the complex plane and their corresponding mapping in the parameter plane. The figure also includes the selection of three controls C_1, C_2, C_3 where control C_3 is the fastest regulator while C_1 would be the slowest.

Using these definitions, the σ -stability regions can be characterized by the following results.

PROPOSITION 4.4

Let $\omega \in \Omega_\sigma$, and let $h, \sigma \in \mathbb{R}_+$ be fixed values. Then, the set $\mathcal{K}_\sigma$ is given by

$$\mathcal{K}_\sigma = \mathcal{K}_{\bar{\sigma}} \cup \mathcal{K}_{\sigma,0}, \quad (4.14)$$

with

$$\begin{aligned} \mathcal{K}_{\sigma,0} &= \left\{ \mathbf{k} \mid k_i = \sigma k_p e^{-h\sigma} + \frac{\sigma}{K} (\sigma T - 1) e^{-h\sigma} \right\}, \\ \mathcal{K}_{\bar{\sigma}} &= \{ \mathbf{k} \mid \mathbf{k} = \mathbf{k}_\sigma(\omega) \}, \end{aligned}$$

where $\mathbf{k}_\sigma(\omega) := [k_{p\sigma}(\omega) \ k_{i\sigma}(\omega)]^\top$, with gains $k_{p\sigma}$ and $k_{i\sigma}$

defined by

$$k_{p\sigma}(\omega) = \frac{\omega(T\omega^2 + 2\sigma - 3\sigma^2)\cos(h\omega)}{K(\sigma\sin(h\omega) - \omega\cos(h\omega))} + \frac{(\omega^2 + \sigma^3T - \sigma^2 - 3T\sigma\omega^2)\sin(h\omega)}{K(\sigma\sin(h\omega) - \omega\cos(h\omega))}, \quad (4.15a)$$

$$k_{i\sigma}(\omega) = \frac{e^{-h\sigma}\omega(2\sigma T - 1)(\omega\sigma^2 + \omega^3)}{K(\sigma\sin(h\omega) - \omega\cos(h\omega))}. \quad (4.15b)$$

Proof. The result is obtained by substituting $s = -\sigma + i\omega$ into (4.6) and proceeding in the same way as in the proof of Proposition 4.2. $\square$

The next result allows deriving a P δ I controller with a specific decay rate and multiple roots.

PROPOSITION

4.5

Let $h \in \mathbb{R}_+$ be a fixed delay. The closed-loop system (4.6) has multiple roots at $s = -\sigma$, $\sigma > \frac{1}{h}$ if and only if the controller's gains $(k_p(\sigma), k_i(\sigma))$ are chosen as:

$$k_p(\sigma) := \frac{-hT\sigma^3 + (3T + h)\sigma^2 - 2\sigma}{Kh\sigma - K}, \quad (4.16a)$$

$$k_i(\sigma) := \frac{(2T\sigma^3 - \sigma^2)e^{-h\sigma}}{Kh\sigma - K}. \quad (4.16b)$$

Proof. The result is obtained in the same way, with the only difference that the substitution is given by $s = -\sigma$. $\square$

Numerical examples

In this section, we apply the results from previous section to different examples of second-order systems using P δ I controller and simulate them via MATLAB [70].

Example 4.2

Consider the following second-order transfer function

$$G(s) = \frac{200}{s^2 + 10s}, \quad (4.17)$$

which follows the structure given in (4.3), where $K = 20$ and $T = 0.1$. In this example, Problem 4.1 is solved using Propositions 4.2 and 4.3 to determine the stability crossing boundaries in terms of (k_p, k_i) , considering a fixed delay $h^* = 0.3$ s. The resulting regions are shown in Figure 4.8.

In the same figure, three controller gains are selected within the region bounded by the CRB and RRB curves.

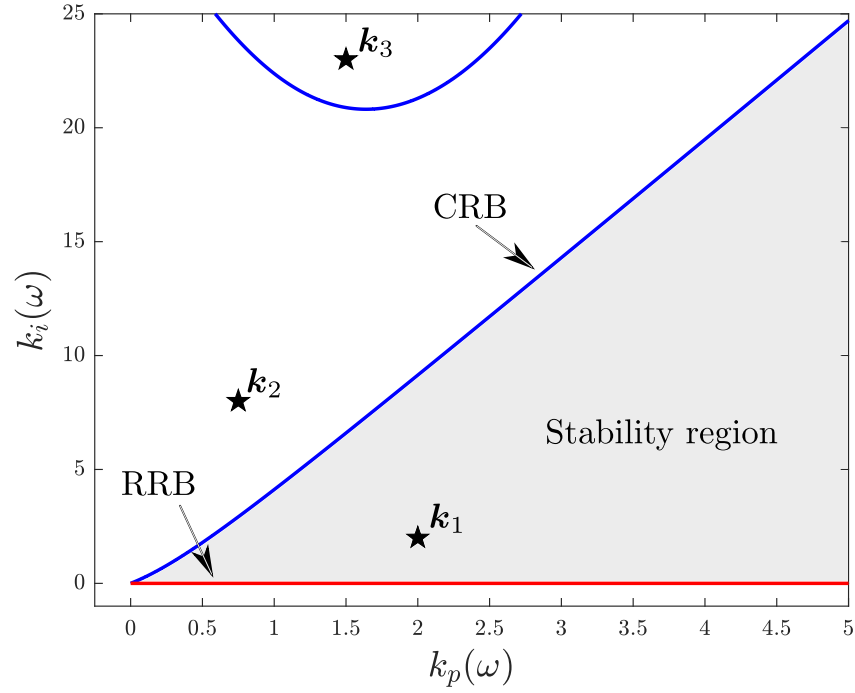


Figure 4.8 / The (k_p, k_i) stability region analysis and selection of different controllers and fixed delay $h^* = 0.125$ s.

The corresponding system responses are presented in Figure 4.9, where one stable response and two unstable cases are observed. These results confirm that Propositions 4.2 and 4.3 can be effectively used to design stabilizing P δ I controllers.

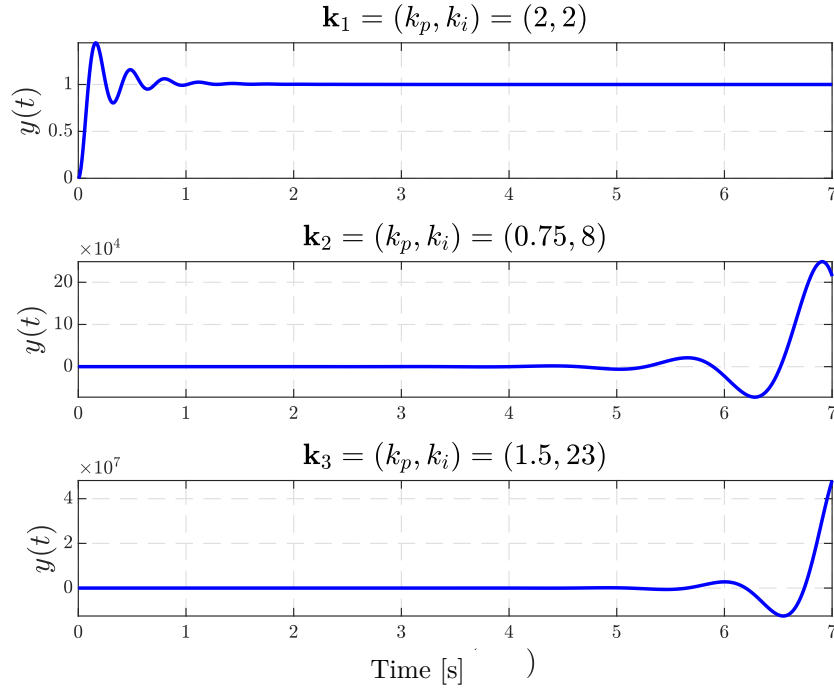


Figure 4.9 / Time step responses with distinct controller gains and fixed delay $h^* = 0.125$ s.

To solve Problem 4.2, the delay is fixed at $h = 0.125$ s, and Proposition 4.4 is used to determine the corresponding σ -stable regions, as shown in Figure 4.14. The values of $\sigma \in \{0, 1.515, 3.030\}$ are considered.

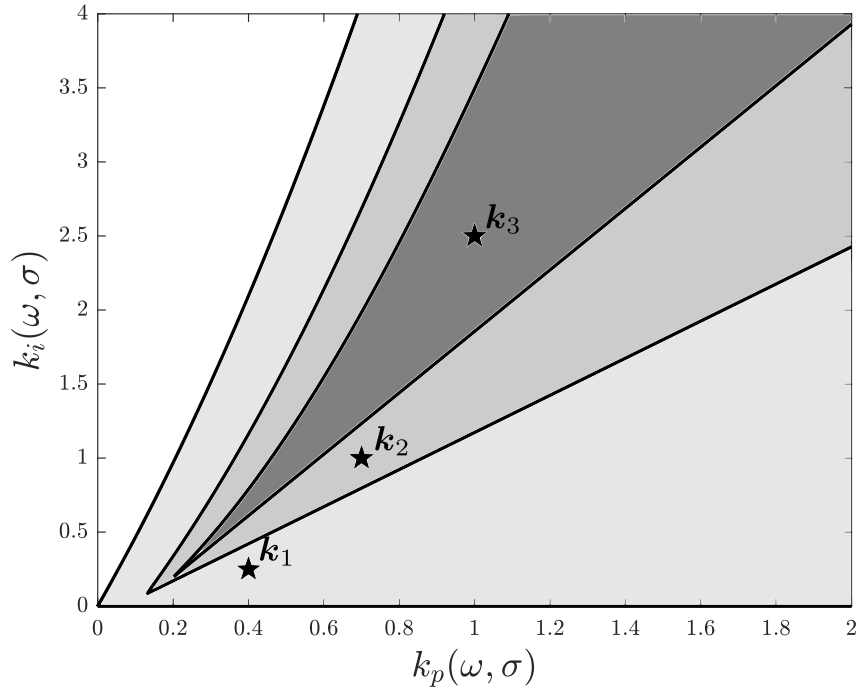


Figure 4.10 / The (k_p, k_i) σ -stability region analysis for system (4.17) and selection of different controllers and fixed delay $h^* = 0.125$ s.

Controller gains are selected within these regions as $\mathbf{k}_1 = (0.4, 0.25)$, $\mathbf{k}_2 = (0.7, 1)$, and $\mathbf{k}_3 = (1, 2.5)$. From the results (see Figure 4.11), it can be observed that increasing σ leads to a faster system response. In particular, the controller $\mathbf{k}_3 = (1, 2.5)$ provides the fastest regulation among the tested cases.

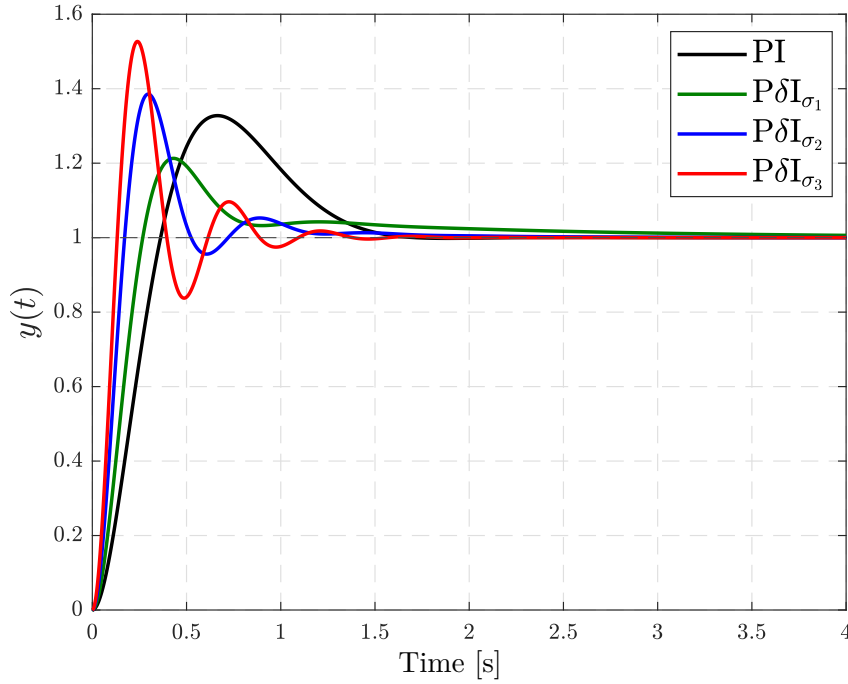


Figure 4.11 / Time step responses with distinct controller gains in σ -stability regions and fixed delay $h^* = 0.125$ s.

From Figure 4.11, it can be observed that a faster response is associated with a larger overshoot (OS). In this example, the OS increases from 32.76% to 52.68%, while the settling time decreases from 1.45 s to 1.02 s.

If the controller gains are selected using Proposition 4.5, together with the value of σ obtained from Proposition 4.4 in an iterative manner, it is possible to search for the maximum allowable σ and, therefore, obtain the fastest achievable regulator. This idea is illustrated in the following example.

Example 4.3: Haptic robotic interface

The dynamics of the haptic system (see Figure 4.12) can be modeled as a decoupled linear time-invariant system. Each joint is described by a mechanical admittance [30], given by the transfer function

$$G(s) = \frac{\Theta(s)}{\Lambda(s)} = \frac{1}{s(ms + b)} = \frac{1}{s(0.0131s + 0.0941)}, \quad (4.18)$$

where $\lambda(t)$ denotes the input torque and $\theta(t)$ represents the angular position. According to the structure in (4.3), the parameters are $K = 10.6269$ and $T = 0.1392$.

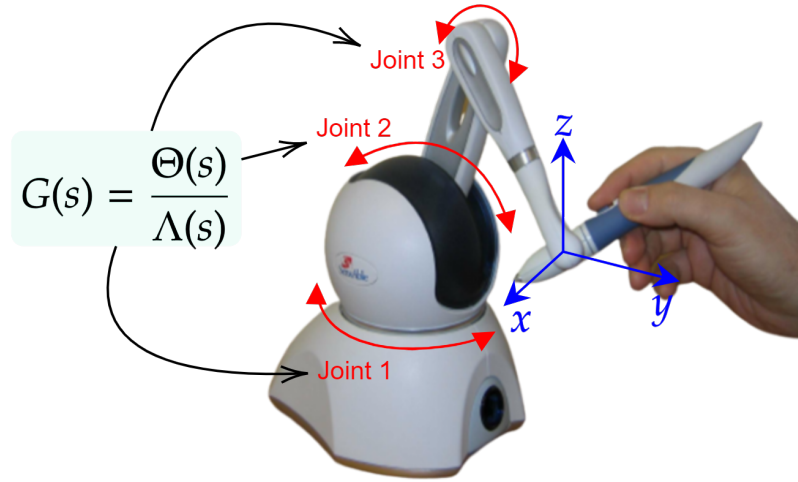


Figure 4.12 / The PHANTOM Omni haptic device.

After applying Propositions 4.4 - 4.5 with $h = 0.1230$ s and $\sigma \in \{0, 1.285, 2.570\}$, three σ -stable regions are obtained, as shown in Figure 4.13. Within these regions, the following controller gains were selected: $\mathbf{k}_1 = (0.6, 0.3)$, $\mathbf{k}_2 = (0.8, 0.8)$, and $\mathbf{k}_3 = (1, 1.5)$.

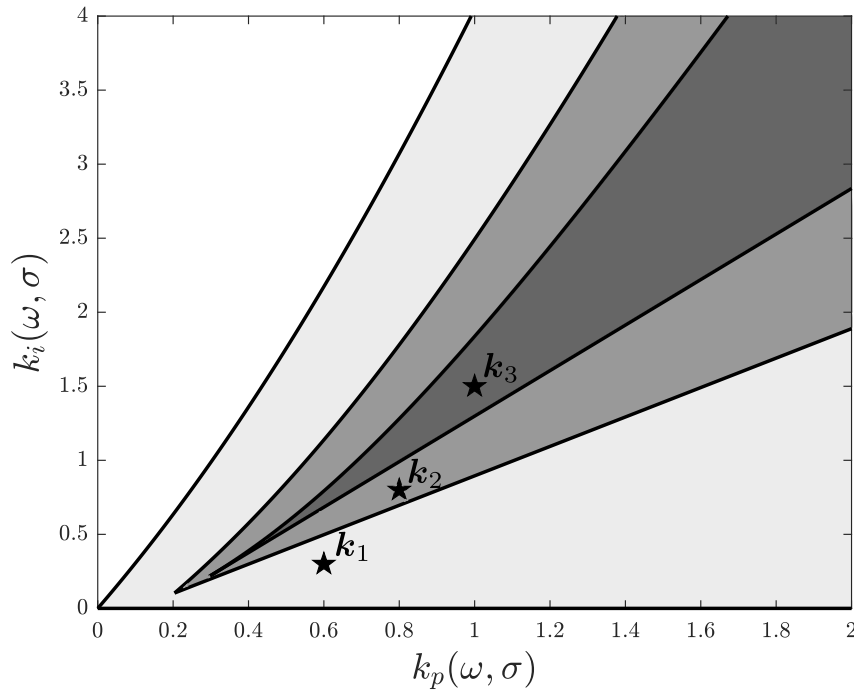


Figure 4.13 / σ -stability regions in the (k_p, k_i) plane for system (4.18), including the selection of different controller gains, with fixed delay $h^* = 0.1715$ s.

Figure 4.14 shows that increasing the σ -stable region results in improved

transient performance, characterized by reduced settling time t_s and rise time t_r . Nevertheless, this enhancement is accompanied by increased oscillations and a higher overshoot.

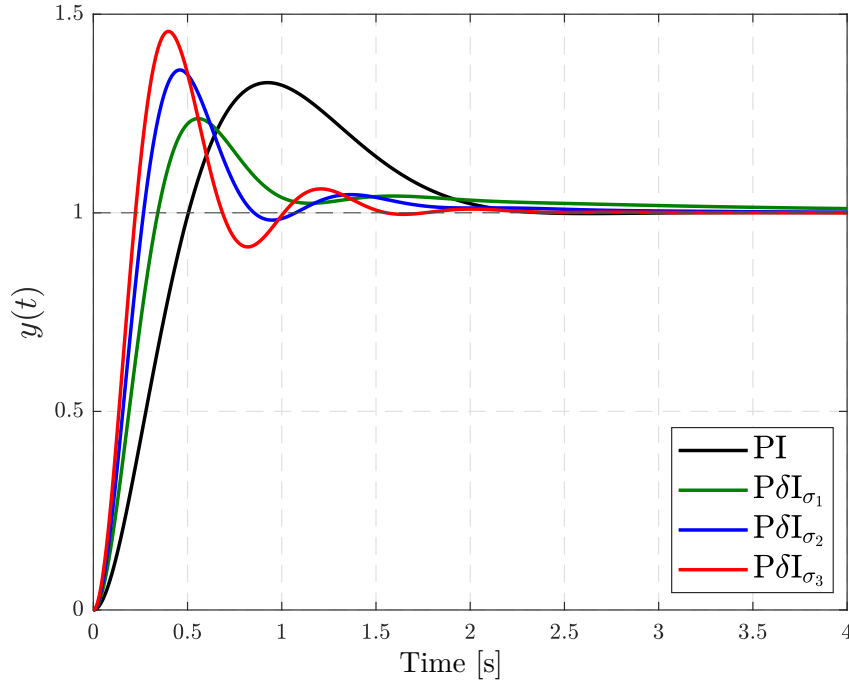


Figure 4.14 / Time step responses corresponding to the selected controller gains within the σ -stable regions, with fixed delay $h^* = 0.1715$ s.

By iteratively applying Proposition 4.4, i.e., by sweeping $\sigma \in [0, 8]$ and $h \in [0, 0.2]$, the maximum exponential decay can be identified, as shown in Figure 4.15. The obtained value is $\sigma_m = 7.69$, which corresponds to the fastest admissible $P\delta I$ controller.

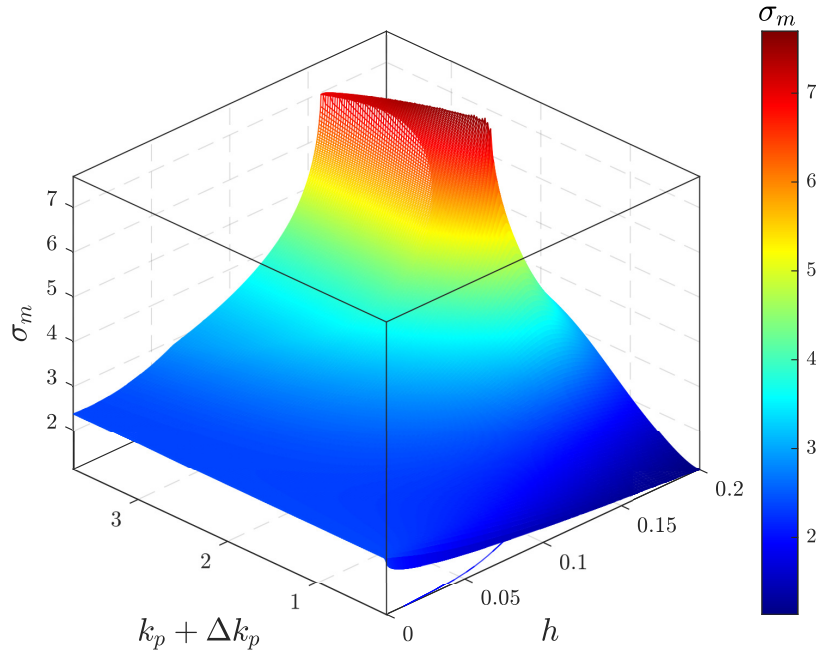


Figure 4.15 / Surface of the parameter triples (k_p, h, σ) illustrating the maximum exponential decay of system (4.18).

To compare the proposed scheme with its classical counterpart, the fastest PI controller is considered. Its gains are obtained through a straightforward application of the Routh–Hurwitz criterion (see, for instance, [71]). In this case, the resulting gains are $\mathbf{k}_{PI} = (0.2253, 0.1798)$.

Similarly, the fastest P δ I controller is given by $\mathbf{k}_{P\delta I} = (2.1253, 4.7292)$ with $h = 0.1715$ s.

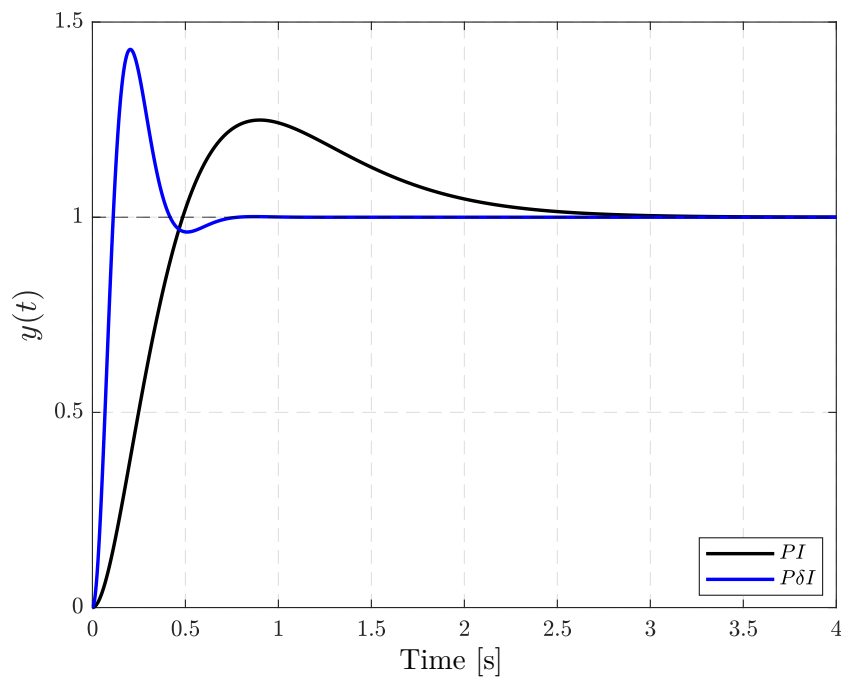


Figure 4.16 / Time-domain comparison of the responses for the PI controller and the PδI controller.

The comparison between both controllers is illustrated in Figure 4.16 and further analyzed using a Kiviat diagram, as shown in Figure 4.17.

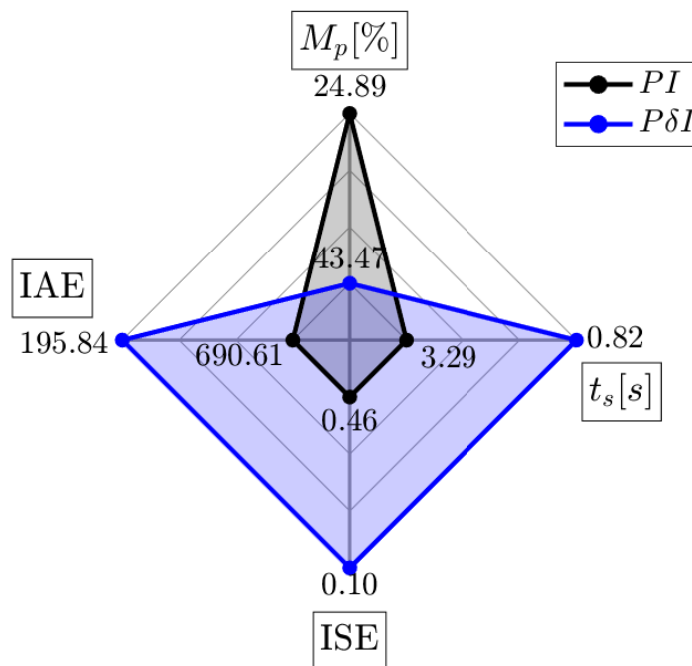


Figure 4.17 / Comparison between the fastest PI controller and the fastest PδI controller.

In conclusion, the fastest P δ I controller outperforms its classical counterpart in terms of the Integral Absolute Error (IAE), Integral Square Error (ISE), and settling time (t_s). For instance, the settling time is reduced from 3.29 s to 0.82 s. However, this improvement is accompanied by an increase in overshoot, from 24.89% to 43.47%.

4.2 Second Order System

This section focuses on the study of classical second-order systems, as they play a fundamental role in automatic control. Understanding how to analyze and solve this type of system, as well as the influence of their parameters on the transient response, is essential in many practical applications. Second-order systems are characterized by having two poles and are typically described by second-order ordinary differential equations. Many real-world systems can be modeled in this form, such as direct current motors, mass–spring–damper systems, and power converters like the boost converter.

Problem formulation

Consider the second-order system defined by the following ordinary differential equation:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2y(t) = K\omega_n^2u(t), \quad (4.19)$$

where $\zeta, \omega_n \in \mathbb{R}$.

To analyze the response of the system in (4.19) to different inputs $u(t)$, the Laplace transform is applied under zero initial conditions. By defining the input–output relationship, the transfer function is obtained as

$$G(s) := \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2}, \quad (4.20)$$

where the parameters K , ζ , and ω_n correspond to the system static gain, damping ratio, and natural frequency, respectively.

To better understand the time-domain behavior of (4.20), the location of its poles is analyzed. These are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (4.21)$$

Since the nature of the poles depends on the value of ζ , four different types of second-order systems can be identified. This classification is summarized in the following definition.

DEFINITION

4.4

(Types of second-order systems)

- Undamped case $\rightarrow \zeta = 0$:

$$s_{1,2} = \pm i\omega_n.$$

- Underdamped case $\rightarrow 0 < \zeta < 1$:

$$s_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2}.$$

- Critically damped case $\rightarrow \zeta = 1$:

$$s_{1,2} = -\omega_n.$$

- Overdamped case $\rightarrow \zeta > 1$:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

Since $\zeta \in \mathbb{R}_+$, it follows that, for the poles of the system in (4.20) to be stable, it is required that $\omega_n \in \mathbb{R}_+$.

Next, a closed-loop configuration is considered, as shown in Figure 4.18.

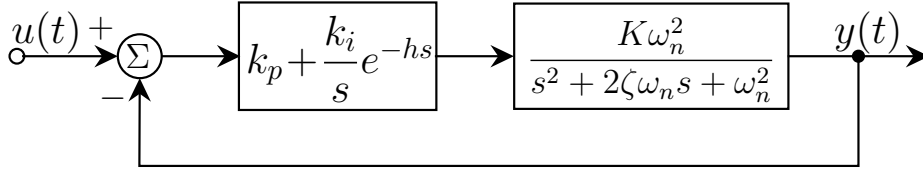


Figure 4.18 / Schematic representation of a standard closed-loop control configuration.

The controller under study is given in (4.2). Under this setup, the resulting closed-loop transfer function is expressed as

$$T(s) = \frac{K\omega_n^2(k_p s + k_i e^{-hs})}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2(Kk_p + 1)s + Kk_i\omega_n^2 e^{-hs}}. \quad (4.22)$$

Analytical approach

Therefore, the stability analysis of the closed-loop system is reduced to the study of the roots of its characteristic quasi-polynomial.

DEFINITION

4.5

Let $\zeta, \omega_n, K, k_p, k_i \in \mathbb{R}$ and $h \in \mathbb{R}_+$. The characteristic quasi-polynomial associated with the closed-loop system in

(4.22) is given by

$$\Delta(s; k_p, k_i, h) = s^3 + 2\zeta\omega_n s^2 + \omega_n^2(Kk_p + 1)s + Kk_i\omega_n^2 e^{-hs}. \quad (4.23)$$

Since the roots of the quasi-polynomial (4.23) vary continuously with respect to continuous changes in the system parameters and the time delay (root continuity) [49], stability regions can be defined in the (k_p, k_i) parameter space for a fixed value of the delay h . To address the proposed objective, the following definitions are recalled.

DEFINITION

4.6

The set of singular frequencies $\Omega \subset \mathbb{R}$ is defined as the set of all ω such that there exists at least one pair (k_p, k_i) for which

$$\Delta(i\omega) = 0. \quad (4.24)$$

That is, Ω contains the frequencies at which the quasi-polynomial admits roots on the imaginary axis.

DEFINITION

4.7

(Stability boundaries) The stability boundaries $\mathcal{K}$ are defined as the set of all controller parameters $\mathbf{k} = (k_p, k_i) \in \mathbb{R}^2$ for which there exists at least one $\omega \in \Omega$ satisfying $\Delta(i\omega) = 0$. Moreover, any point $\mathbf{k} \in \mathcal{K}$ is referred to as a crossing point.

Stability crossing curves

Based on the previous definitions, the following results are obtained.

PROPOSITION

4.6

Let $\omega \in \mathbb{R}_+$. Then, ω is a singular frequency if and only if there exists a controller gain vector $\mathbf{k}(\omega, h) := [k_p(\omega, h) \ k_i(\omega, h)]^\top$ such that

$$k_p(\omega, h) = \frac{2\zeta\omega\omega_n \tan(h\omega) + \omega^2 - \omega_n^2}{K\omega_n^2}, \quad (4.25a)$$

$$k_i(\omega, h) = \frac{2\zeta\omega^2 \sec(h\omega)}{K\omega_n}. \quad (4.25b)$$

Proof. The result is obtained by substituting $s = i\omega$ into (4.23) and separating real and imaginary parts. $\square$

Remark 4.2. Proposition 4.6 considers only positive frequencies. This is justified by the fact that

$$\Delta(-i\omega) = \overline{\Delta(i\omega)}, \quad (4.26) \quad \downarrow$$

↳ which implies that the roots are symmetric with respect to the real axis. See [72] for further details. ┘

PROPOSITION
4.7

The location of the roots of the quasi-polynomial (4.23) at $s = 0$ satisfies

$$k_p \in (-\infty, +\infty), \quad (4.27a)$$

$$k_i = 0. \quad (4.27b)$$

Proof. The result follows directly by substituting $s = 0$ into (4.23). $\square$

σ –stability

As in the second-order integrator case, the σ -stability tuning methodology will also be employed. The loss of σ -stability occurs when the quasi-polynomial (4.23) has roots at $s = -\sigma$ or a pair of purely imaginary roots at $s = -\sigma \pm i\omega$. In the following, the roots on the imaginary axis, which act as candidates for stability or instability transitions, are characterized.

PROPOSITION
4.8

For a fixed delay value h^* , the quasi-polynomial (4.23) partitions the (k_p, k_i) plane through the parametric equations (4.28a)–(4.28b).

$$k_p(\omega, \sigma, h) = \frac{\sin(h\omega)(-2\zeta\sigma^2\omega_n + 2\zeta\omega^2\omega_n + \sigma^3 + \sigma(\omega_n^2 - 3\omega^2))}{K\omega_n^2(\omega \cos(h\omega) - \sigma \sin(h\omega))} + \frac{\omega \cos(h\omega)(4\zeta\sigma\omega_n - 3\sigma^2 + \omega^2 - \omega_n^2)}{K\omega_n^2(\omega \cos(h\omega) - \sigma \sin(h\omega))}, \quad (4.28a)$$

$$k_i(\omega, \sigma, h) = -\frac{2\omega e^{-h\sigma}(\sigma^2 + \omega^2)(\sigma - \zeta\omega_n)}{K\omega_n^2(\omega \cos(h\omega) - \sigma \sin(h\omega))}. \quad (4.28b)$$

Furthermore, the value of σ determines the σ -stable regions of the system (4.22).

Proof. The objective is to obtain the parametric representation in the (k_p, k_i) plane, following the approach in [8]. For this purpose, the parametric expressions $k_p(\omega, \sigma, \tau)$ and $k_i(\omega, \sigma, \tau)$ are derived.

First, the change of variable $s = \hat{s} - \sigma$ is introduced to enforce σ -stability. Then, by setting $\hat{s} = i\omega$, the characteristic quasi-polynomial is evaluated at

$$\Delta(s)|_{s=i\omega-\sigma} = 0. \quad (4.29)$$

Since this substitution leads to a quasi-polynomial with complex coefficients, the ┘

condition above is equivalent to imposing

$$\begin{cases} \Re\{\Delta(s)|_{s=i\omega-\sigma}\} = 0, \\ \Im\{\Delta(s)|_{s=i\omega-\sigma}\} = 0. \end{cases} \quad (4.30)$$

This results in a system of two equations with two unknowns, from which k_p and k_i can be obtained. $\square$

The previous result defines a parametric curve for $0 < \omega < +\infty$. It remains to characterize the case $\omega = 0$, which yields a relation depending only on σ . In this case, the corresponding curve represents the set of controller gains for which at least one root lies on the line L_σ .

PROPOSITION

4.9

Considering the parameters of the quasi-polynomial (4.23) given in Definition 4.5, the following parametric expressions are obtained in the limit $\omega \rightarrow 0$:

$$k_{pr} \in (-\infty, +\infty), \quad (4.31a)$$

$$k_{ir}(\sigma, h) = \frac{\sigma e^{-h\sigma}(Kk_{pr}\omega_n^2 - 2\zeta\sigma\omega_n + \sigma^2 + \omega_n^2)}{K\omega_n^2}. \quad (4.31b)$$

Proof. The result is obtained by following the same procedure as in Proposition 4.8, with the difference that the substitution $s = -\sigma$ is considered. $\square$

On the other hand, an additional parametric curve can be defined, corresponding to the vertex where the CRB and RRB curves intersect. This curve represents the set of controller gains for which a multiple root is located at $s = -\sigma$.

PROPOSITION

4.10

The quasi-polynomial (4.23) admits multiple roots for the controller gains (k_p, k_i) given by

$$k_{p0}(\sigma, h) = \frac{\sigma^2(2h\zeta\omega_n + 3) - \sigma\omega_n(h\omega_n + 4\zeta) - h\sigma^3 + \omega_n^2}{K\omega_n^2(h\sigma - 1)}, \quad (4.32a)$$

$$k_{i0}(\sigma, h) = \frac{2\sigma^2 e^{-h\sigma}(\sigma - \zeta\omega_n)}{K\omega_n^2(h\sigma - 1)}. \quad (4.32b)$$

Proof. The result follows by taking the limit of the parametric expressions as $\omega \rightarrow 0$, that is,

$$k_{p0}(\sigma, h) = \lim_{\omega \rightarrow 0} k_p(\omega, \sigma, h), \quad (4.33a)$$

$$k_{i0}(\sigma, h) = \lim_{\omega \rightarrow 0} k_i(\omega, \sigma, h). \quad (4.33b)$$

According to previous works such as [35] [32] [73] [31], the maximum exponential decay is typically associated with a single point in the parameter space. This situation arises when the closed-loop system has a multiple real root. However, in this work, a different approach is proposed to determine the maximum spectral abscissa associated with multiple real roots. In contrast to existing results, this maximum does not necessarily correspond to the highest achievable exponential decay, and instead leads to a region in the parameter space rather than a unique point.

In [74], a method was proposed to determine the maximum exponential decay $\sigma_{\max}$ based on a second-derivative approach applied to the quasi-polynomial (?). This procedure allows the elimination of the exponential term, leading to a system of three equations with three unknowns (k_p, k_i, σ) .

PROPOSITION
4.11

If the system has a triple root at $\omega = 0$ and $\sigma = \sigma_{\max}$, then the following system of equations must be satisfied:

$$\Delta(s, \sigma) = (s - \sigma)^3 + 2\zeta\omega_n(s - \sigma)^2 + \omega_n^2(1 + Kk_p)(s - \sigma) + Kk_i\omega_n^2e^{-(s-\sigma)h} = 0, \quad (4.34a)$$

$$\Delta'(s, \sigma) = 3(s - \sigma)^2 + 4\zeta\omega_n(s - \sigma) + \omega_n^2(1 + Kk_p) - Kk_i\omega_n^2he^{-(s-\sigma)h} = 0, \quad (4.34b)$$

$$\Delta''(s, \sigma) = 6(s - \sigma) + 4\zeta\omega_n + Kk_i\omega_n^2h^2e^{-(s-\sigma)h} = 0. \quad (4.34c)$$

In this work, a methodology is proposed to determine $\sigma_{\max}$ associated with multiple real roots, based on the slope matching of the curves defined by (4.28) and (4.32). That is,

PROPOSITION
4.12

The maximum exponential decay $\sigma_{\max}$ of the closed-loop system (4.22) is attained when the slopes of the curves defined by (4.28) and (4.32) are equal.

Proof. Consider the closed-loop system (4.22). It was observed that, as the σ -stable regions are compacted, they do not converge to a single point but rather to a region. To determine $\sigma_{\max}$, a geometric approach based on the analysis of the slopes of the defining curves is employed.

As the regions contract with increasing σ , a limiting value $\sigma_{\max}$ is reached. At this stage, the curves intersect, and instead of further contraction, a new region emerges, indicating an expansion (see Figure 4.19).

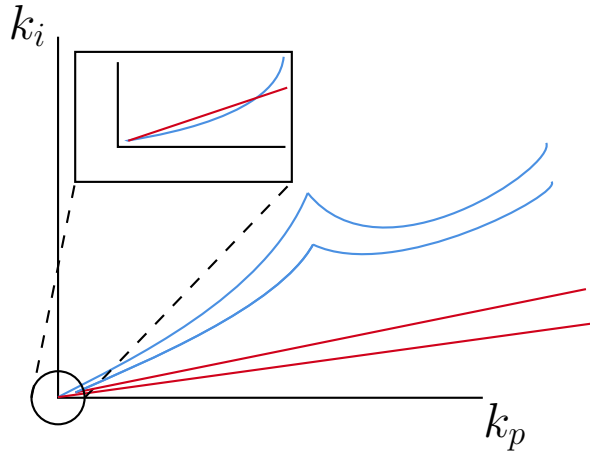


Figure 4.19 / Stability boundaries in the (k_p, k_i) plane.

The expressions for the slopes are obtained using the implicit function theorem, yielding

$$m_\sigma(\sigma, h) = \frac{dk_i}{dk_p} = - \frac{\frac{\partial \Delta}{\partial k_p}}{\frac{\partial \Delta}{\partial k_i}}, \quad (4.35a)$$

$$m_\omega(\omega, \sigma, h) = \frac{dk_i}{dk_p} = \frac{\frac{\partial k_i}{\partial \omega}}{\frac{\partial \omega}{\partial k_p}}. \quad (4.35b)$$

As shown in Figure 4.19, the curves intersect at the initial vertex and again when $\sigma = \sigma_{\max}$. This condition can be determined analytically by enforcing equality of the slopes, that is,

$$m_\sigma - m_\omega = 0. \quad (4.36)$$

□

Numerical examples

Example 4.4: DC Motor

Consider an electromechanical system as shown in Figure 4.20.

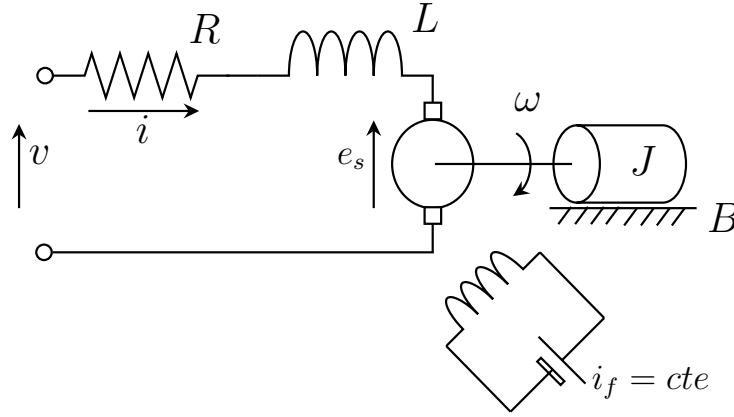


Figure 4.20 / Electromechanical system.

The armature of the direct current (DC) motor is modeled as a series connection of a constant resistance R and an inductance L , which represents the armature winding inductance. The input voltage v corresponds to the applied armature voltage, while the field current i_f is assumed to be constant.

After performing the electrical and mechanical analysis of the system, the resulting transfer function relating the angular velocity $\Omega(s)$ to the input voltage $V(s)$ is given by

$$\frac{\Omega(s)}{V(s)} = \frac{\frac{K_m}{LJ}}{s^2 + \left(\frac{R}{L} + \frac{B}{J}\right)s + \frac{RB + K_m K_a}{LJ}}. \quad (4.37)$$

By establishing a correspondence with the general second-order system in (4.20), the parameters K , ω_n , and ζ can be expressed in terms of the physical parameters of the DC motor.

For this study, parameter values reported in the literature are adopted, as summarized in Table 4.1.

Parameter	Symbol	Value	Unit
Moment of inertia	J	0.00040095	$kg\ m^2$
Viscous friction coefficient	B	0.00223579	$N\ m\ s$
Back EMF constant	K_a	0.23351629	$V/(rad/s)$
Motor torque constant	K_m	0.23351629	$N\ m/A$
Armature resistance	R	3.4	Ω
Armature inductance	L	0.0015	H

Table 4.1 / DC motor parameters.

From these values, the equivalent second-order parameters are obtained as

$$K = 3.75842, \quad \omega_n = 321.414, \quad \zeta = 3.53476.$$

Thus, the model in (4.38) can be rewritten in standard second-order form as

$$\frac{\Omega(s)}{V(s)} = \frac{1.0331 \times 10^5}{s^2 + 2.2722 \times 10^3 s + 1.0331 \times 10^5}. \quad (4.38)$$

This corresponds to an open-loop stable system. For this case, the maximum spectral abscissa is found to be $\sigma_{\max} = 10.01$.

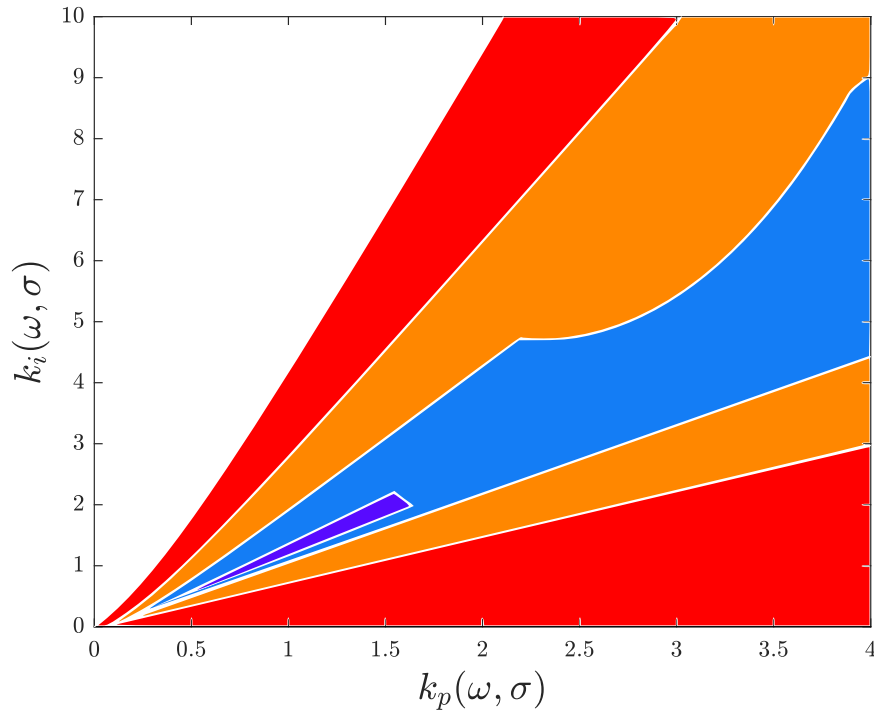


Figure 4.21 / Stability regions in the (k_p, k_i) plane for $h = 0.1$ s and $\sigma \in \{0, 2.5, 5, 7.5, \sigma_{\max}\}$

As shown in Figure 4.21, the value of $\sigma_{\max}$ associated with multiple real roots does not converge to a single point (i.e., a unique pair of gains) in the parameter plane, but rather to a region (a set of gain pairs). Subsequently, different gain pairs were selected within the corresponding stability regions, and the resulting closed-loop responses are presented in Figure 4.22.

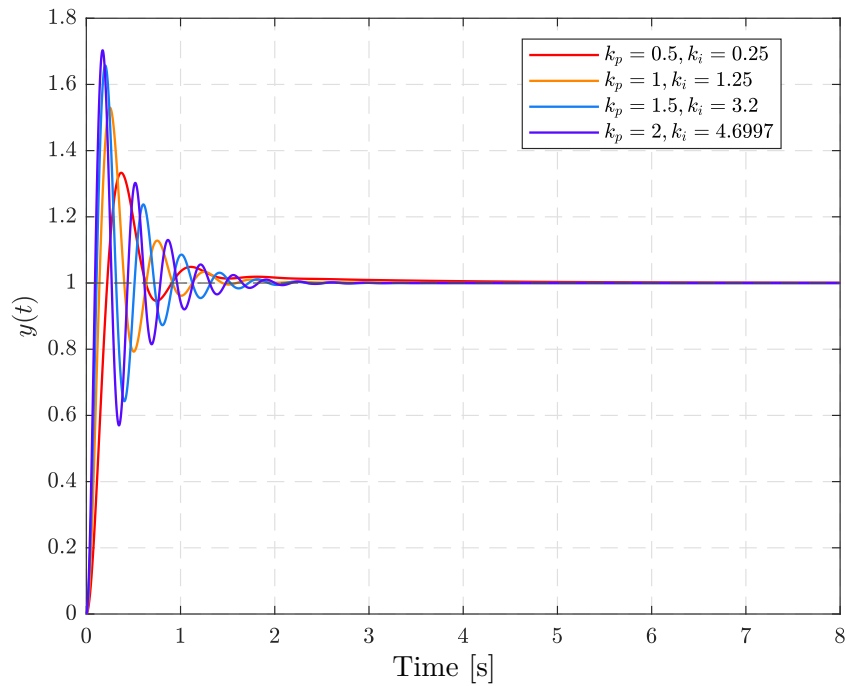


Figure 4.22 / Closed-loop responses with fixed controller gains and delay.

In a different scenario, the controller gains were kept fixed while the delay was varied to evaluate its effect on the system performance. In particular, the influence of the delay on the transient response was analyzed in terms of overshoot and settling time. The corresponding responses are shown in [Figure 4.23](#).

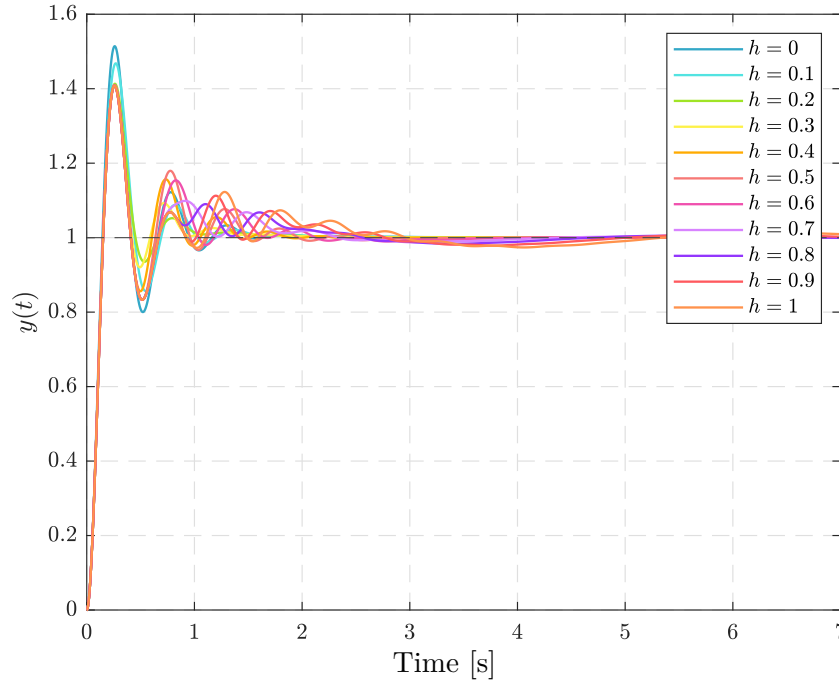


Figure 4.23 / Closed-loop responses with fixed controller gains and varying delay.

4.3 Second Order Non-Minimum Phase System

Problem formulation

Consider the class of LTI, SISO, second-order systems described by the transfer function

$$G(s) \triangleq \frac{P(s)}{Q(s)} := \frac{K\omega_n^2(1 - zs)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad (4.39)$$

where $K \in \mathbb{R}_+$, $\zeta \in \mathbb{R}$, $\omega_n \in \mathbb{R} \setminus \{0\}$, and $z \in \overline{\mathbb{R}}_+$. The plant (4.39) is considered in closed loop with a modified classical PI controller, referred to as the P δ I controller, whose frequency-domain representation is given by (4.2).

The resulting closed-loop system is formed by the interconnection of plant (4.39) and controller (4.2) under negative unit feedback, as shown in Figure 4.24.

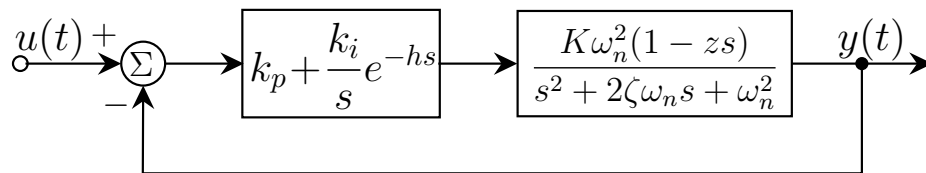


Figure 4.24 / General block representation of a conventional closed-loop control architecture.

The main problems addressed in this section are formulated as follows:

Problem 4.3: Stability conditions

Given an LTI SISO system with transfer function (4.39), derive analytical conditions on the parameters (k_p, k_i, h) under which the P δ I controller in (4.2) stabilizes the closed-loop system (see Figure 4.24). In particular, analyze the cases where either h or k_p is fixed, and provide explicit stability conditions.

Problem 4.4: Optimize exponential decay

For the second-order system (4.39) in closed loop with the P δ I controller (4.2) (see Figure 4.24), determine the gains $(k_p^*, k_i^*, h^*) \in \mathbb{R}^2 \times \mathbb{R}_+$ that minimize the spectral abscissa of the closed-loop system.

Problem 4.5: Optimize delay

Consider the system (4.39) with a stabilizing PI controller defined by the gains (k_p^*, k_i^*) (3.4). Analyze whether the spectral abscissa can be improved by introducing the delay h , and if so, determine the corresponding optimal value h^* .

To illustrate the main ideas behind the previously stated problems, two examples are presented. These examples highlight two key effects: improving the transient performance of the response and inducing a stabilizing effect in the closed-loop system.

Example 4.5: DC–DC boost converter

Consider the second-order non-minimum phase (NMP) system corresponding to a DC–DC boost converter [75], given by

$$G(s) = \frac{1.04 \times 10^{10}(1 - 12 \times 10^{-5}s)}{s^2 + 4166.58s + 3.47 \times 10^7}, \quad (4.40)$$

in closed loop with a standard PI controller (3.4).

The open-loop poles of the system are located at $s = -2.0833 \times 10^3 \pm i 5.5119 \times 10^3$. By applying the classical Routh–Hurwitz (RH) criterion [76], the MED is obtained as $\sigma_{m\text{PI}} = 2165.97$ (see Definition 2.20).

The MED characterizes the fastest achievable response, as it corresponds to placing the rightmost closed-loop pole as far as possible to the left in the complex plane. Therefore, the main objective is to shift this pole leftward while preserving stability. Figure 4.25 illustrates this effect, showing the decay of the error signal together with an exponential envelope defined by σ_m , which corresponds to the real part of the dominant pole.

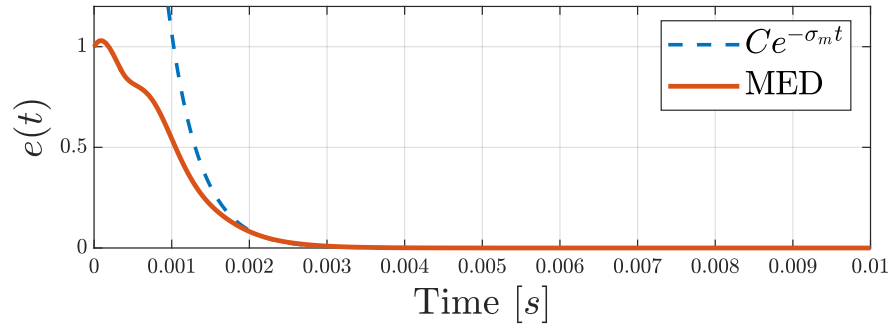
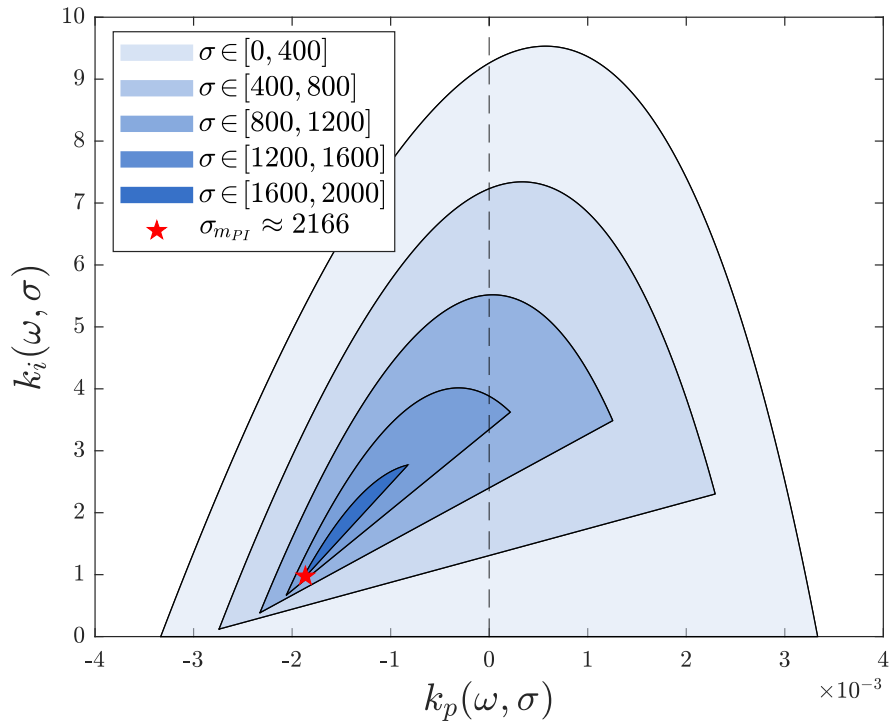
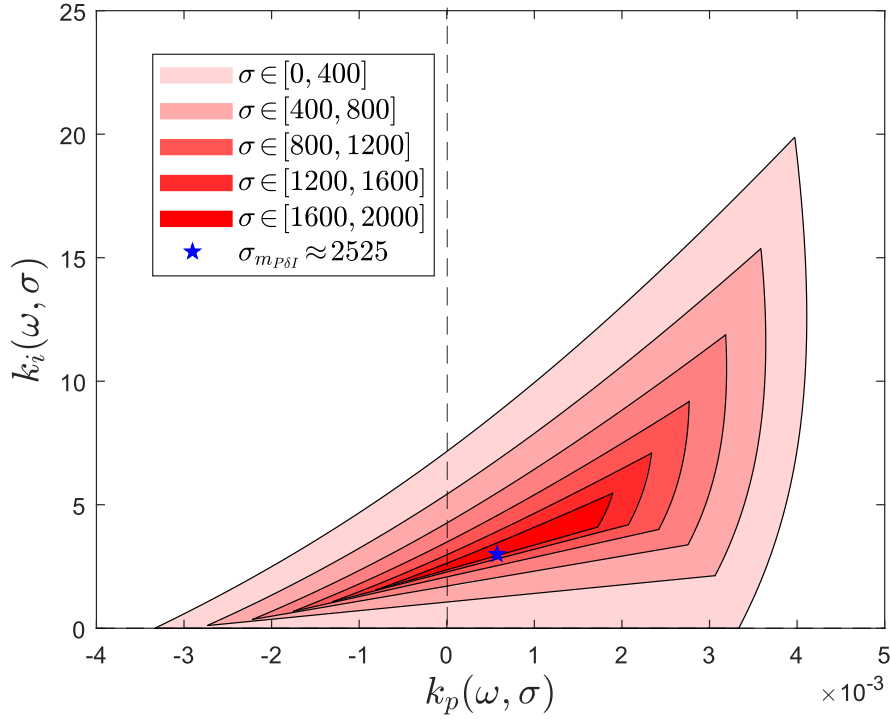


Figure 4.25 / Illustration of the relationship between the rightmost characteristic root and the decay rate of the error signal.

It is worth noting that a limitation of the classical PI controller is the introduction of an additional unstable zero when attempting to achieve this performance, which leads to conservative results. In this example, the MED is attained with $k_p = -1.9 \times 10^{-3}$ and $k_i = 0.9755$ (see Figure 4.26a).



(a) (k_p, k_i) -plane for the PI controller.



(b) (k_p, k_i) -plane for the $P\delta I$ controller with $h \neq 0$.

Figure 4.27 / Stability regions for the PI and $P\delta I$ controllers.

In this section, a delayed PI controller, referred to as the $P\delta I$ controller, is considered. This approach introduces an intentional delay h as an additional tuning parameter. As will be shown, this extra degree of freedom allows for improved σ -stability, enabling the rightmost closed-loop poles to be shifted further to the left without introducing unstable zeros in $\mathbb{C}_+$.

To illustrate these advantages, the behavior of system (4.40) is analyzed in closed loop with the controller $P\delta I(4.2)$. By selecting the parameters $k_p = 0.00057$, $k_i = 2.9952$, and $h = 3.44 \times 10^{-4}$ s, a significant improvement in σ -stability is achieved, with $\sigma_{m_{P\delta I}} = 2524.7$ (see Figure 4.26b).

This result highlights a clear performance advantage over the classical PI controller. A comparison of the corresponding time responses is shown in Figure 4.28.

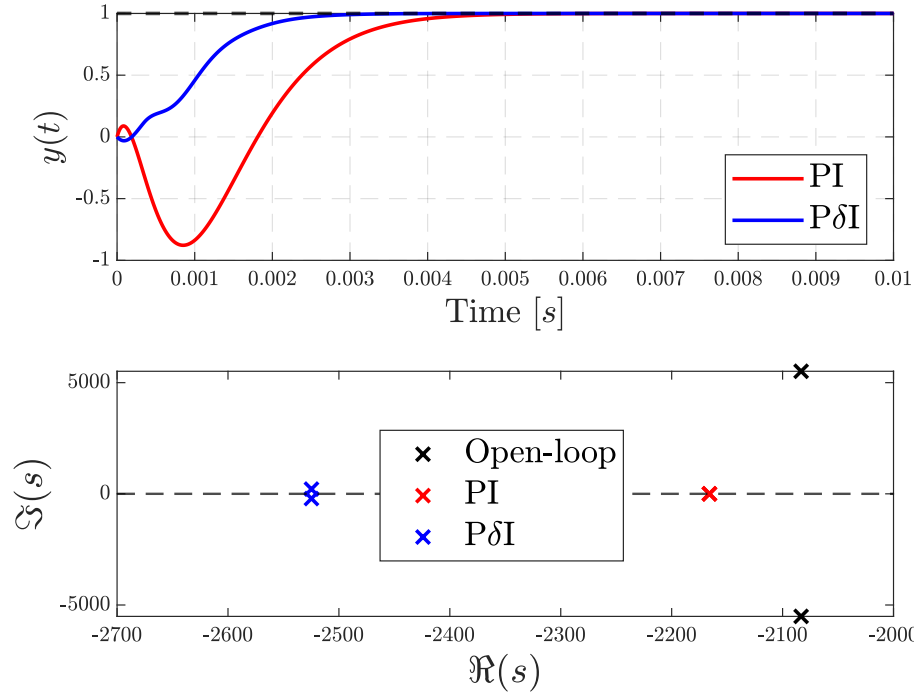


Figure 4.28 / Time responses and root-locus comparison between optimal PI and PδI controllers.

Analytical approach

According to Figure 4.24, the closed-loop characteristic function $\Delta : \mathbb{C} \times \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{C}$ is given by

$$\Delta(s; \mathbf{k}, h) \triangleq s^3 + (2\zeta\omega_n - Kk_p\omega_n^2)s^2 + \omega_n^2(1 + Kk_p)s + Kk_i\omega_n^2(1 - zs)e^{-hs}, \quad (4.41)$$

where $\mathbf{k} = [k_p \ k_i]^\top$. The first objective in this work is to determine controller parameters that guarantee exponential stability of the closed-loop system. To this end, we introduce the following definition.

Remark 4.3. Throughout this section, the terms *exponentially stable quasi-polynomials* and *stable quasi-polynomials* are used interchangeably. $\square$

PROPOSITION 4.13

Let the closed-loop characteristic function $\Delta(s; \mathbf{k}, h)$ be given by (4.41). Then, $k_i > 0$ is a necessary condition for exponential stability.

Proof. The proof follows by contradiction. Assume that there exists a stabilizing controller with $k_i < 0$. Then, all the roots of $\Delta(s; \mathbf{k}, h)$ must lie in the left-half plane.

Evaluating at the origin yields $\Delta(0; \mathbf{k}, h) = K\omega_n^2 k_i < 0$. On the other hand, as $s \rightarrow +\infty$, it follows that $\Delta(s; \mathbf{k}, h) \rightarrow +\infty$. By the *Intermediate Value Theorem* $\curvearrowright$

↳ [77], there exists at least one $r \in \mathbb{R}_+$ such that $\Delta(r; \mathbf{k}, h) = 0$.

This implies the existence of a real positive root, which contradicts the stability assumption. Therefore, $k_i < 0$ cannot stabilize the system, and it must hold that $k_i > 0$. $\square$

Stability crossing curves

In this section, a graphical methodology is proposed to determine the stability regions in the controller parameter plane. To identify these regions, it is important to recall that, due to the continuity of the characteristic roots of retarded quasi-polynomials with respect to the system parameters [64], stability changes can only occur when roots cross the imaginary axis.

Such crossings can take place in two distinct ways. First, a root may cross the imaginary axis at a finite nonzero frequency, i.e., at $s = i\omega$. This case is referred to as the CRB. Second, a root may cross through the origin, i.e., at $s = 0$, which defines the RRB (see Figure 4.29).

To characterize these boundaries, the sets $\mathcal{T}_\omega$ (or $\mathcal{K}_\omega$) and $\mathcal{T}_0$ (or $\mathcal{K}_0$) are introduced to denote the CRB and RRB, respectively, such that $\mathcal{T} = \mathcal{T}_\omega \cup \mathcal{T}_0$ (or $\mathcal{K} = \mathcal{K}_\omega \cup \mathcal{K}_0$).

In this framework, the mapping from the complex plane to the parameter space is obtained analytically, allowing the construction of stability regions.

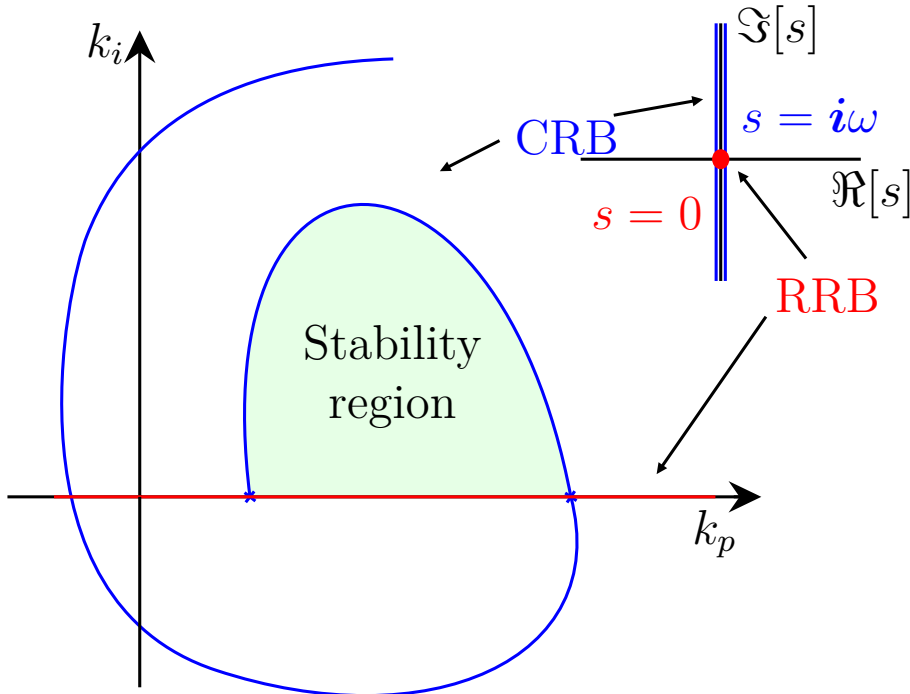


Figure 4.29 / Mapping of stability curves to the parameter plane.

To characterize the CRB curve, the following set is introduced:

DEFINITION

4.8

The *frequency crossing set* $\Omega \subset \mathbb{R}$ is defined as the set of all ω for which there exists at least one triplet $(k_p, k_i, h) \in \mathbb{R}^2 \times \mathbb{R}_+$ such that

$$\Delta(i\omega; \mathbf{k}, h) = 0. \quad (4.42)$$

Remark 4.4. In the following, any $\omega \in \Omega$ will be referred to as a *singular frequency*. $\lrcorner$

Remark 4.5. It follows directly that

$$\Delta(-i\omega; \mathbf{k}, h) = \overline{\Delta(i\omega; \mathbf{k}, h)}. \quad (4.43)$$

Therefore, in the remainder of this work, it is sufficient to consider only nonnegative frequencies, i.e., $\Omega \subset \mathbb{R}_+$. $\lrcorner$

DEFINITION

4.9

For a fixed $h^* > 0$, the *stability crossing curves*, denoted by $\mathcal{K}$, are defined as the set of all parameters $\mathbf{k} = (k_p, k_i) \in \mathbb{R}^2$ for which there exists at least one $\omega \in \Omega$ such that

$$\Delta(i\omega; \mathbf{k}, h^*) = 0. \quad (4.44)$$

Similarly, for a fixed $k_p^* \in \mathbb{R}$, the set $\mathcal{T}$ denotes the stability crossing curves defined in the (h, k_i) space, i.e., the set of all parameters $\boldsymbol{\kappa} = (h, k_i) \in \mathbb{R}_+ \times \mathbb{R}$ for which there exists at least one $\omega \in \Omega$ such that

$$\Delta(i\omega; k_p^*, \boldsymbol{\kappa}) = 0. \quad (4.45)$$

Any point $\mathbf{k} \in \mathcal{K}$ (or $\boldsymbol{\kappa} \in \mathcal{T}$) is referred to as a *crossing point*.

PROPOSITION

4.14

Let G be the open-loop system, and assume it is not an *even system* (see Remark 4.12). Let $h^* > 0$ be a fixed delay in the control scheme (4.2). Then, the set $\mathbf{k}_0 \in \mathbb{R}^2$ defined by

$$\mathbf{k}_0 \triangleq \left\{ \begin{bmatrix} k_p \\ 0 \end{bmatrix} \in \mathbb{R}^2 \mid k_p \in \mathbb{R} \right\}, \quad (4.46)$$

belongs to the stability crossing curve. Moreover, this set corresponds to a crossing at the origin of the complex plane.

Proof. Evaluating (4.41) at $s = 0$ yields

$$\Delta(0; \mathbf{k}, h^*) = 0 \implies k_i P(0) = 0.$$

Since $P(0) = K\omega_n^2 \neq 0$ (see (4.39)), it follows that $k_i = 0$, which completes the proof. $\square$

Remark 4.6. From the above result, it follows that $\mathcal{T}_0 \equiv \mathbf{k}_0$. $\lrcorner$

PROPOSITION
4.15

Assume that G is not an even system (see Remark 4.12). Let $h^* > 0$ be fixed, and define $\Omega \triangleq \bigcup_{\ell} \Omega_{\ell}$, with $\ell \in \overline{\mathbb{N}}$, where

$$\Omega_{\ell} \triangleq \left\{ \omega \in \mathbb{R}_+ \mid \omega \in \left(\frac{\delta_{\ell}\pi}{2h^*} (2\ell - 1), \frac{\pi}{2h^*} (2\ell + 1) \right) \right\},$$

and $\delta_{\ell} := \text{sign}(\ell)$. Then, $\omega \in \Omega$ is a crossing frequency if and only if $\mathbf{k}(\omega) \triangleq [k_p(\omega) \ k_i(\omega)]^{\top}$ is given by

$$k_p(\omega) \triangleq -\Re \left\{ \frac{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega i}{K\omega_n^2(1 - z\omega i)} \right\} + \Im \left\{ \frac{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega i}{K\omega_n^2(1 - z\omega i)} \right\} \tan(\omega h^*), \quad (4.47a)$$

$$k_i(\omega) \triangleq \omega \Im \left\{ \frac{\omega_n^2 - \omega^2 + 2\zeta\omega_n\omega i}{K\omega_n^2(1 - z\omega i)} \right\} \sec(\omega h^*), \quad (4.47b)$$

which defines a crossing point $\mathbf{k}(\omega) \in \mathcal{K}_{\omega}$.

Proof. Consider the characteristic equation (4.41). For any $\mathbf{k} \in \mathcal{K}_{\omega}$, the following conditions must hold:

$$\Re\{\Delta(i\omega; \mathbf{k}, h^*)\} = 0, \quad (4.48a)$$

$$\Im\{\Delta(i\omega; \mathbf{k}, h^*)\} = 0. \quad (4.48b)$$

Solving (4.48) for k_p and k_i yields (4.47). Furthermore, from (4.47), it follows that $\mathbf{k}(\omega)$ is real-valued if and only if $\omega \neq \frac{\pi}{2h^*} (2n - 1)$, for $n \in \mathbb{Z}$. Hence, $\mathbf{k}(\omega)$ is real for all $\omega \in \Omega$. $\square$

PROPOSITION
4.16

Consider the curve $\mathcal{K}_{\omega}$, and let $\mathbf{k}(\omega) \in \mathcal{K}_{\omega}$. Then, the vertex at which the curve originates is given by

$$\mathbf{k}_v \triangleq \begin{bmatrix} -\frac{1}{K} \\ 0 \end{bmatrix}. \quad (4.49)$$

Proof. The result follows directly by evaluating the limit

$$\lim_{\omega \rightarrow 0} \mathbf{k}(\omega).$$

□

Next, by fixing $k_p^* \in \mathbb{R}$, the following results characterize the set $\mathcal{T} \subset \mathbb{R}_+ \times \mathbb{R}$ in the (h, k_i) -plane.

PROPOSITION
4.17

Let $k_p^* \in \mathbb{R}$ be fixed. Then, the set $\boldsymbol{\kappa}_0 \equiv \mathcal{T}_0$, with $\boldsymbol{\kappa}_0 \in \mathbb{R}_+ \times \mathbb{R}$, is defined as

$$\boldsymbol{\kappa}_0 \triangleq \left\{ \begin{bmatrix} h_0 \\ 0 \end{bmatrix} \mid h_0 \in \mathbb{R}_+ \right\}, \quad (4.50)$$

and belongs to the stability crossing curve. Moreover, this set corresponds to a crossing through the origin of the complex plane.

Proof. The proof follows the same arguments as in Proposition 4.14. □

PROPOSITION
4.18

Let $k_p^* \in \mathbb{R}$ be fixed. Then, $\omega \in \Omega$ is a crossing frequency if and only if $\boldsymbol{\kappa}(\omega) \triangleq [h(\omega) \ k_i(\omega)]^\top$, where

$$h(\omega) \triangleq -\frac{1}{\omega} \left(\arctan(\omega z) + \text{Arg}(\hat{\alpha}(\omega) + i\hat{\beta}(\omega)) - (2m + \epsilon)\pi \right), \quad (4.51a)$$

$$k_i(\omega) \triangleq \pm \frac{\sqrt{\hat{\alpha}^2(\omega) + \hat{\beta}^2(\omega)}}{K\omega_n^2 \sqrt{1 + z^2\omega^2}}, \quad (4.51b)$$

defines a crossing point $\boldsymbol{\kappa}(\omega) \in \mathcal{T}_\omega$, with

$$\hat{\alpha}(\omega) \triangleq \omega^2 \omega_n (Kz k_p^* \omega_n - 2\zeta),$$

$$\hat{\beta}(\omega) \triangleq \omega_n^2 (K\omega k_p^* + \omega) - \omega^3,$$

$m \in \mathbb{Z}$, and

$$\epsilon \triangleq \begin{cases} 1, & \text{if } k_i \geq 0, \\ 0, & \text{if } k_i < 0. \end{cases}$$

Proof. The proof follows the same procedure as in Proposition 4.15. □

Propositions 4.13–4.18 play a central role in characterizing the stability crossing curves. These curves partition the parameter space into regions with a constant number of unstable roots, i.e., roots located in $\mathbb{C}_+$. Consequently, when crossing a stability boundary, at least one root continuously moves from $\mathbb{C}_+$ to $\mathbb{C}_-$, or vice

versa.

Crossing direction characterization

Therefore, it is essential to develop tools that allow the identification of stability regions, that is, regions in the parameter space containing only stabilizing pairs (k_p, k_i) . To this end, the following results provide a characterization of the crossing direction of the characteristic roots.

PROPOSITION

4.19

Let $h^* > 0$ be fixed. A root of (4.39) crosses the imaginary axis through the origin from $\mathbb{C}_-$ to $\mathbb{C}_+$ as $\mathbf{k}$ traverses $\mathcal{K}_0$ in the increasing direction of k_i , provided that the intersection point lies to the left of $\mathbf{k}_0^{(i)} \in \mathcal{K}_0$, defined as

$$\mathbf{k}_0^{(i)} := \begin{bmatrix} -\frac{1}{K} \\ 0 \end{bmatrix}. \quad (4.52)$$

Conversely, the crossing occurs from $\mathbb{C}_+$ to $\mathbb{C}_-$ if the intersection lies to the right of $\mathbf{k}_0^{(i)}$.

PROPOSITION

4.20

Let $h^* > 0$ be fixed, and let $\mathbf{k}^* = [k_p(\omega^*) \ k_i(\omega^*)]^\top \in \mathbb{R}^2$ be a point on the stability crossing curve $\mathcal{K}$ corresponding to the crossing frequency $\omega^* \in \Omega$. Then, a pair of simple roots of (4.39) crosses from $\mathbb{C}_-$ to $\mathbb{C}_+$ as $\mathbf{k}$ traverses $\mathcal{K}$ at $\mathbf{k}^*$ in the direction of increasing k_i if

$$\vartheta(\omega^*; \mathbf{k}^*, h^*) > 0, \quad (4.53)$$

where

$$\vartheta(\omega; \mathbf{k}, h) \triangleq \text{sign}(k_i) \left[\left(h + \frac{z^2}{1 + z^2 \omega^2} \right) + \Re \left\{ \frac{\varphi'(\mathbf{i}\omega, k_p)}{\varphi(\mathbf{i}\omega, k_p)} \right\} \right],$$

with $\varphi(s, k_p) \triangleq Q(s) + k_p P(s)$ and $\varphi'(s, k_p) = \frac{d\varphi}{ds}$. The crossing direction is reversed if (4.53) does not hold.

Proof. Let $h^* > 0$ be fixed. By assumption, $s^* = \mathbf{i}\omega^*$ (with $\omega^* > 0$) is a simple root of Δ for $\mathbf{k}^* \in \mathbb{R}^2$, which implies that $\varphi(\mathbf{i}\omega^*, k_p^*) \neq 0$, and thus ϑ is well-defined.

As $\mathbf{k}$ crosses $\mathcal{K}$, the root $s(\mathbf{k})$ evolves continuously. Hence, the direction of crossing is determined by the sign of

$$\Re \left\{ \frac{ds}{dk_i} \right\}.$$

↴

◁ In particular, the crossing occurs from $\mathbb{C}_-$ to $\mathbb{C}_+$ if

$$\Re\left\{\frac{ds}{dk_i}\right\} > 0,$$

and in the opposite direction otherwise.

By applying the *Implicit Function Theorem* [65], one obtains

$$\frac{ds}{dk_i} = - \frac{\frac{\partial \Delta(s^*; \mathbf{k})}{\partial k_i} \Big|_{\mathbf{k}=\mathbf{k}^*}}{\frac{\partial \Delta(s; \mathbf{k}^*)}{\partial s} \Big|_{s=s^*}}.$$

The result follows by observing that the sign of this expression coincides with $\vartheta(\omega; \mathbf{k}, h)$. $\square$

Remark 4.7. Proposition 4.20 remains valid when the stability crossing curves are described by $\mathcal{T}$ instead of $\mathcal{K}$, i.e., when k_p^* is fixed and the parameter vector $\boldsymbol{\kappa}$ is considered. The proof follows analogously. $\lrcorner$

As previously mentioned, the stability of a quasi-polynomial is determined by the location of its roots in the complex plane. However, computing these roots is not straightforward and typically requires specialized tools such as QpMR [78] or TDS-Control [79].

Stability index determination

In contrast, this work adopts a graphical approach to assess stability by tracking the crossing directions of the roots. This allows the number of unstable poles, i.e., those located in $\mathbb{C}_+$, to be determined without explicitly computing the roots.

In the following, the goal is to determine the *stability index* of a given region, that is, for a given pair of controller gains, to compute the number of unstable roots associated with the region where they lie.

To this end, let $\mathbf{k}^\dagger = [k_p^\dagger \ k_i^\dagger]^T$ be an arbitrary point in the (k_p, k_i) -plane such that $\mathbf{k}^\dagger \notin \mathcal{K}$. Let η_0 denote the number of unstable open-loop roots, and let η denote the number of roots in the right-half plane (RHP) corresponding to the region containing $\mathbf{k}^\dagger$.

Consider the line segment connecting the origin $\mathcal{O}$ to $\mathbf{k}^\dagger$, denoted by $\overline{\mathcal{O}\mathbf{k}^\dagger}$. Each intersection between $\overline{\mathcal{O}\mathbf{k}^\dagger}$ and the curve $\mathcal{K}_\omega$ occurs at a point of the form $\varepsilon \mathbf{k}^\dagger$, where $\varepsilon \in (0, 1)$.

Define the set Ω_s as the set of all $\omega \in \Omega$ such that $\mathbf{k}^\dagger$ intersects $\mathcal{K}_\omega$. This occurs if and only if

$$k_p^\dagger k_i(\omega) - k_i^\dagger k_p(\omega) = 0, \quad (4.54)$$

together with the condition

$$0 < \frac{k_p(\omega)}{k_p^\dagger} < 1, \quad 0 < \frac{k_i(\omega)}{k_i^\dagger} < 1. \quad (4.55)$$

Thus, $\Omega_s := \{\omega_1, \omega_2, \dots, \omega_\ell\}$, where ℓ denotes the number of intersections.

To account for the direction of each crossing, define the indicator function $\mathcal{I}_\varepsilon : \mathbb{R}_+ \rightarrow \{-1, 0, 1\}$ as

$$\mathcal{I}_\varepsilon(\omega_r) := \begin{cases} \vartheta(\omega_r; \mathbf{k}^*, h^*), & \text{if } \varepsilon \in (0, 1), \\ 0, & \text{otherwise,} \end{cases} \quad (4.56)$$

for $r \in \{1, 2, \dots, \ell\}$, where $\vartheta(\omega; \mathbf{k}, h)$ is defined in (4.53).

PROPOSITION
4.21

Let $\mathbf{k}^\dagger = [k_p^\dagger \ k_i^\dagger]^T \notin \mathcal{K}$. Then, for all $\mathbf{k}$ in the same region as $\mathbf{k}^\dagger$, the number of unstable roots $\eta \in \overline{\mathbb{N}}$ is given by

$$\eta = \eta_0 + \frac{1 - \text{sign}(k_i^\dagger)}{2} + 2 \sum_{\omega \in \Omega_s} \mathcal{I}_\varepsilon(\omega). \quad (4.57)$$

Proof. Fix $\mathbf{k}^\dagger$ and consider the associated sets Ω_s and $\mathcal{I}_\varepsilon$. From Proposition 4.16, the origin $\mathcal{O}$ does not lie on the curve $\mathcal{K}_\omega$. Indeed, for $[0, 0]^\top \in \mathcal{K}_\omega$, it would be necessary that $\omega_n = 0$, which is excluded by assumption. Hence, η_0 corresponds directly to the number of unstable roots of $Q(s)$.

Each intersection between $\overline{\mathcal{O}\mathbf{k}^\dagger}$ and $\mathcal{K}_\omega$ corresponds to a pair of complex conjugate roots crossing the imaginary axis. The direction of each crossing is determined by the sign of $\Re\{ds/dk_i\}$, which is captured by $\mathcal{I}_\varepsilon$.

Therefore, the total number of unstable roots is obtained by adding the initial count η_0 and the contributions of each crossing, yielding (4.57). $\square$

σ -stability

In the stability analysis of dynamical systems, the *spectral abscissa* of characteristic quasi-polynomials plays a key role. This real-valued quantity determines the rate at which solutions decay in stable systems and, conversely, indicates the presence of unstable modes. On this note, we focus on minimizing the spectral abscissa by analyzing the so-called σ -stability.

The main idea is to analyze the relative stability of the system with respect to a shifted vertical line, referred to as the σ -axis, which is achieved by applying the change of variable $s \rightarrow s - \sigma$. This transformation enables a systematic tuning of the controller gains to enforce a desired decay rate.

Such an approach enables the identification of parameter regions for which the closed-loop characteristic roots satisfy $\Re\{s\} < -\sigma$. To assess σ -stability, it is

necessary to analyze the roots located on the shifted imaginary axis $s = -\sigma \pm i\omega$ for a given triplet (k_p, k_i, h) .

In this context, let $\mathcal{K}_\sigma$ (or $\mathcal{T}_\sigma$) denote the set of parameters (k_p, k_i, h) for which the quasi-polynomial (4.39) admits at least one root on the vertical line

$$\mathbf{L}_\sigma \triangleq \{s \in \mathbb{C} \mid s = -\sigma + i\omega, \omega \in \mathbb{R}_+\},$$

(see Remark 4.5). This line will be referred to as the σ -axis.

Next, by incorporating Definition 2.20, Propositions 4.14–4.18 can be reformulated as follows.

PROPOSITION
4.22

Let $h^* > 0$ be fixed and $\sigma > 0$. Define $\Omega_\sigma \triangleq \bigcup_\ell \Omega_{\sigma,\ell}$, where

$$\Omega_{\sigma,\ell} \triangleq \left\{ \omega \in \mathbb{R}_+ \mid \omega \in \Omega_\ell \wedge \tan(h^*\omega) \neq \frac{\omega}{\sigma} \right\}.$$

Then, the set $\mathcal{K}_\sigma$ is given by

$$\mathcal{K}_\sigma \triangleq \mathcal{K}_{\sigma,0} \cup \mathcal{K}_{\sigma,\omega}, \quad (4.58)$$

where

$$\mathcal{K}_{\sigma,0} \triangleq \left\{ \mathbf{k}_{\sigma,0} \in \mathbb{R}^2 : e^{h^*\sigma} k_i = \sigma \left(\frac{\sigma^2 + 2\zeta\omega_n\sigma + \omega_n^2}{K\omega_n^2(1+z\sigma)} + k_p \right) \right\}, \quad (4.59)$$

$$\mathcal{K}_{\sigma,\omega} \triangleq \left\{ \mathbf{k}_{\sigma,\omega} \in \mathbb{R}^2 : \mathbf{k}_{\sigma,\omega} = [k_p(\omega, \sigma) \ k_i(\omega, \sigma)]^\top \right\}, \quad (4.60)$$

and $k_p(\omega, \sigma), k_i(\omega, \sigma) \in \mathbb{R}$ are defined as

$$k_p(\omega, \sigma) \triangleq \frac{\tan(h^*\omega) \Re\{F(\xi)\} + \Im\{F(\xi)\}}{\sigma \tan(h^*\omega) - \omega}, \quad (4.61a)$$

$$k_i(\omega, \sigma) \triangleq \frac{e^{-h^*\sigma} (\omega \Re\{F(\xi)\} + \sigma \Im\{F(\xi)\}) \sec(\omega h^*)}{\sigma \tan(h^*\omega) - \omega}, \quad (4.61b)$$

where $\xi \triangleq -\sigma + i\omega$ and $F(s) \triangleq \frac{s}{G(s)}$.

Proof. The proof follows the same procedure as in Propositions 4.14–4.15, under the change of variable $s = -\sigma + i\omega$. $\square$

PROPOSITION
4.23

Let $\sigma > 0$ and $h^* > 0$ be fixed, and let $\mathbf{k}_{\sigma,\omega}(\omega)$ be defined by (4.61). Then, the departure point $\mathbf{k}_{\sigma,v} \in \mathbb{R}^2$ of the curve

$\mathbf{k}_{\sigma,\omega}$ is given by

$$\mathbf{k}_{\sigma,v} \triangleq \begin{bmatrix} \frac{h^* F(-\sigma) - F'(-\sigma)}{h^* \sigma - 1} \\ \frac{e^{-h^* \sigma} (F(-\sigma) - \sigma F'(-\sigma))}{h^* \sigma - 1} \end{bmatrix}, \quad (4.62)$$

where $F(s) \triangleq \frac{s}{G(s)}$, and $F'(s) \triangleq \frac{d}{ds} F(-s)$.

Proof. To prove the result, it suffices to show that the following limit holds:

$$\lim_{\omega \rightarrow 0} \mathbf{k}_{\sigma,\omega} = \mathbf{k}_{\sigma,v}.$$

Before computing this limit, observe the following key relations:

$$\lim_{\omega \rightarrow 0} \Re\{F(\xi)\} = F(-\sigma) \quad \text{and} \quad \lim_{\omega \rightarrow 0} \frac{\Im\{F(\xi)\}}{\omega} = F'(-\sigma). \quad (4.63)$$

With these observations in mind, we proceed to compute the limit. Note that:

$$\begin{aligned} \lim_{\omega \rightarrow 0} \mathbf{k}_{\sigma,\omega} &= \lim_{\omega \rightarrow 0} \begin{bmatrix} \frac{(\omega h^* + \mathcal{O}(|\omega h^*|^3)) F(-\sigma)}{\sigma(h^* \omega + \mathcal{O}(|\omega h^*|^3)) - \omega} \\ \frac{e^{-h^* \sigma} \left(\omega F(-\sigma) \left(1 + \frac{1}{2} \omega^2 h^{*2} + \mathcal{O}(|\omega h^*|^4) \right) + \sigma \Im\{F(\xi)\} \right)}{\sigma(h^* \omega + \mathcal{O}(|\omega h^*|^3)) - \omega} \end{bmatrix} \\ &\equiv \begin{bmatrix} \frac{h^* F(-\sigma) - F'(-\sigma)}{h^* \sigma - 1} \\ \frac{e^{-h^* \sigma} (F(-\sigma) - \sigma F'(-\sigma))}{h^* \sigma - 1} \end{bmatrix}. \end{aligned}$$

This computation confirms that:

$$\lim_{\omega \rightarrow 0} \mathbf{k}_{\sigma,\omega} = \mathbf{k}_{\sigma,v},$$

and thus, the result is proved. $\square$

In some situations, it is convenient to fix the gain k_p and derive the stability boundaries in terms of h and k_i . With this in mind, the following result is obtained.

PROPOSITION

Let $k_p^* \in \mathbb{R}$ be fixed. Then, $\omega \in \Omega$ is a crossing frequency if

and only if $\kappa(\omega, \sigma) \triangleq [h(\omega, \sigma) \ k_i(\omega, \sigma)]^\top$ is given by

$$h(\omega, \sigma) \triangleq -\frac{1}{\omega} \left(\tan^{-1} \left(\frac{\omega z}{1 + \sigma z} \right) - \text{Arg} \left\{ \left(\check{\alpha}(\omega, \sigma) + i\omega \check{\beta}(\omega, \sigma) \right) \right\} - (2m + \epsilon)\pi \right), \quad (4.64a)$$

$$k_i(\omega, \sigma) \triangleq \pm e^{-\sigma h} \frac{\sqrt{\check{\alpha}^2(\omega, \sigma) + \omega^2 \check{\beta}^2(\omega, \sigma)}}{K\omega_n^2 \sqrt{(1 + \sigma z)^2 + z^2 \omega^2}}, \quad (4.64b)$$

where $m \in \mathbb{Z}$,

$$\epsilon \triangleq \begin{cases} 1, & \text{if } k_i \geq 0, \\ 0, & \text{if } k_i < 0, \end{cases}$$

and

$$\begin{aligned} \check{\alpha}(\omega, \sigma) &\triangleq -\sigma^3 + (2\zeta\omega_n - Kk_p^*\omega_n^2 z)\sigma^2 \\ &\quad + (3\omega^2 - \omega_n^2 - Kk_p^*\omega_n^2)\sigma \\ &\quad + Kk_p^*\omega_n^2 z\omega^2 - 2\zeta\omega_n\omega^2, \\ \check{\beta}(\omega, \sigma) &\triangleq 3\sigma^2 + (2Kk_p^*\omega_n^2 z - 4\zeta\omega_n)\sigma + Kk_p^*\omega_n^2 + \omega_n^2 - \omega^2. \end{aligned}$$

Furthermore, the curve

$$k_i(\sigma) \triangleq \frac{e^{-h\sigma} K\omega_n^2 (1 + z\sigma)}{\sigma^3 - (2\zeta\omega_n - Kk_p^*\omega_n^2 z)\sigma^2 + (\omega_n^2 + Kk_p^*\omega_n^2)\sigma} \quad (4.65)$$

belongs to $\mathcal{T}_{\sigma,0}$.

Proof. The result follows by applying the change of variable $s = -\sigma + i\omega$ in (4.41) and proceeding as in Propositions 4.17–4.18. $\square$

Remark 4.8. As discussed at the beginning of this section, Propositions 4.22–4.23 enable a systematic tuning of the P δ I controller while ensuring that the dominant poles remain strictly to the left of the σ -axis. $\lrcorner$

Optimized PI with delay

A direct application of the root-crossing direction in the proposed control design is the performance optimization of a PI controller in the presence of a time delay. Specifically, once a stabilizing PI controller has been obtained, an intentional delay h is introduced in the integral action and subsequently tuned to improve the spectral abscissa, without modifying the original gains (k_p, k_i) .

The proposed methodology provides a systematic condition to tune the delay while preserving the stability properties of the delay-free PI controller. This condition serves as both a theoretical and practical guideline: first, to guarantee

closed-loop stability, and second, to enhance the transient response by increasing the convergence rate.

This condition is based on the root-crossing direction of the closed-loop system, which allows the delay to be adjusted in such a way that root trajectories evolve according to the predicted behavior while remaining within stability boundaries.

PROPOSITION
4.25

Let $\mathbf{k}^* = [k_p^* \ k_i^*]^\top$ define stabilizing controller gains for the characteristic function $\Delta(s; \mathbf{k}, 0)$. Let $s_d \in \mathbb{C}$ denote its rightmost root, such that s_d is simple. If, for a sufficiently small $h > 0$, the following inequality holds:

$$\Re\{\vartheta_h(s_d)\} < 0, \quad (4.66)$$

where

$$\vartheta_h(s_d) := \frac{K k_i^* \omega_n^2 s_d (1 - s_d z) e^{-h s_d}}{\Psi}, \quad (4.67)$$

and

$$\begin{aligned} \Psi := & 3s_d^2 + (4\zeta\omega_n - 2K k_p^* \omega_n^2 z) s_d \\ & + \omega_n^2 (1 + K k_p^*) - K k_i^* \omega_n^2 (z + h - h z s_d) e^{-h s_d} \end{aligned}$$

then the spectral abscissa $\alpha(\Delta(s; \mathbf{k}^*, h))$ is minimized.

Proof. The result follows from continuity arguments. It is clear that if the spectral abscissa is locally decreasing concerning h , the following condition must be satisfied:

$$\text{sign} \left\{ \Re \left\{ \frac{ds}{dh} \Big|_{s=s_d, h=h^*} \right\} \right\} < 0,$$

which is equivalent to (4.66) and satisfied $\alpha(\Delta(s_d; \mathbf{k}^*, h^*)) < \alpha(\Delta(s; \mathbf{k}^*, 0))$. This observation ends the proof. $\square$

Remark 4.9. Proposition 4.25 provides sufficient conditions under which the P δ I controller can improve the decay rate achieved by a PI controller. These conditions are not necessary, and a complete characterization would require additional analysis. $\lrcorner$

Based on the previous result, an algorithm is proposed to determine the delay h that optimizes the spectral abscissa. The procedure begins by selecting a small delay h^* and verifying that condition (4.67) holds for the rightmost root s_d . The delay is then gradually increased until (4.67) becomes zero, at which point the process is stopped and the optimal delay h^* is identified. The complete procedure is summarized in Algorithm 2.

Algorithm 2 Determination of optimal delay.

```

1: Define the system  $G(s)$ .
2: Define the gains of the PI controller  $\mathbf{k}^*$ .
3: Let  $h^* > 0$  compute  $s_d = \max(\Re\{\Delta(s; \mathbf{k}^*, h^*) = 0\})$ .
4: if  $\vartheta_h(s_d) < 0$  then
5:   Perform initial check on  $\vartheta_h(s_d)$ .
6:   while  $\vartheta_h(s_d) \neq 0$  do
7:     Increment  $h^*$ .
8:     Compute  $s_d = \max(\Re\{\Delta(s; \mathbf{k}^*, h^*) = 0\})$ .
9:     Recompute  $\vartheta_h(s_d)$ .
10:    if  $\vartheta_h(s_d) = 0$  then
11:      break.
12:    end if
13:  end while
14:  Save the corresponding delay value  $h^* \triangleq h^*$ .
15: end if

```

Remark 4.10. In the Algorithm 2 note that, on the one hand, s_d is a simple root and, on the other hand, ϑ_h captures the direction in which s_d moves as the delay parameter h varies. Since $\vartheta_h > 0$, the spectral abscissa α shifts towards the right, and s_d corresponds to a local minimum. Therefore, no delay adjustment is needed. $\lrcorner$

Remark 4.11. Proposition 4.25 is restricted to the case of simple roots, since the presence of multiple roots would require a bifurcation analysis, for which the implicit function theorem is no longer applicable. A possible extension to handle multiple roots would involve incorporating higher-order derivatives into the analysis. This direction is left for future work. $\lrcorner$

Delay margin

The delay margin characterizes the robustness of the closed-loop system with respect to delay variations. More precisely, it is defined as the maximum admissible value of the delay h for which the system remains stable. In this sense, the delay margin is defined as follows:

DEFINITION
4.10

Consider the closed-loop characteristic function $\Delta(s; \mathbf{k}, h)$ given in (4.41) and its spectral abscissa $\alpha(\Delta(s; \mathbf{k}, h))$. Assume that the controller gains $\tilde{\mathbf{k}} = [\tilde{k}_p \ \tilde{k}_i]^\top$ correspond to a stabilizing controller. Then the *delay margin* of the closed-loop system is defined by

$$\bar{h} \triangleq \sup\{\beta \in \mathbb{R}_+ : \alpha(\Delta) < 0 \quad \forall h \in [\tilde{h}, \beta)\},$$

where $\tilde{h}$ is the corresponding delay parameter of the controller.

With this definition in mind, consider the function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined as

$$\Phi(\omega) = \omega^6 + \alpha_1 \omega^4 + \alpha_2 \omega^2 + \alpha_3, \quad (4.68)$$

where

$$\alpha_1 \triangleq \omega_n^2 (K k_p \omega_n z - 2\zeta)^2 - 2K k_p \omega_n^2 - 2\omega_n^2, \quad (4.69a)$$

$$\alpha_2 \triangleq -K^2 k_i^2 \omega_n^4 z^2 + K^2 k_p^2 \omega_n^4 + 2K k_p \omega_n^4 + \omega_n^4, \quad (4.69b)$$

$$\alpha_3 \triangleq -K^2 k_i^2 \omega_n^4. \quad (4.69c)$$

Next, define the set $\Lambda \subset \overline{\mathbb{R}}_+^2$ as the set of all pairs (ω, h) satisfying

$$\Phi(\omega) = 0, \quad (4.70a)$$

$$\cos(h\omega) = -\Re \left\{ \frac{\omega(\omega_n^2(-1 + K k_p(-1 + i\omega z)) - 2i\zeta\omega_n\omega + \omega^2)}{K k_i \omega_n^2(\omega z + i)} \right\}, \quad (4.70b)$$

$$\sin(h\omega) = \Im \left\{ \frac{\omega(\omega_n^2(-1 + K k_p(-1 + i\omega z)) - 2i\zeta\omega_n\omega + \omega^2)}{K k_i \omega_n^2(\omega z + i)} \right\}. \quad (4.70c)$$

The following result holds.

PROPOSITION

4.26

Let $(\tilde{k}_p, \tilde{k}_i, \tilde{h})$ be a stabilizing set of control parameters. Then, the closed-loop system remains stable for all $h \in (\tilde{h}, \bar{h})$, where $\bar{h}$ is defined as

$$\bar{h} \triangleq \min_{(\omega, h) \in \Lambda} h. \quad (4.71)$$

Moreover, for $h = \bar{h}$, a characteristic root lies on the imaginary axis at $s = i\omega^*$, where $(\omega^*, \bar{h}) \in \Lambda$.

Proof. From $\Delta(i\omega; \mathbf{k}, h) = 0$, it follows that

$$\omega(\omega_n^2(K k_p(\omega z + i) + i) - 2\zeta\omega_n\omega - i\omega^2) = K k_i \omega_n^2 e^{-i\omega h} (1 - i\omega z). \quad (4.72)$$

Taking magnitudes on both sides yields (4.70a), while equating arguments leads to (4.70b) and (4.70c).

Finally, since $\Delta(s; \mathbf{k}, \tilde{h})$ is stable and the characteristic roots depend continuously on h , it follows that all $h \in [\tilde{h}, \bar{h})$ preserve stability. At $h = \bar{h}$, a root reaches the imaginary axis, completing the proof. $\square$

Numerical examples

To demonstrate the applicability and effectiveness of the proposed results, this section presents four representative numerical examples, including two theoretical systems—one unstable and one stable—an oscillatory system, and a practical case involving a power electronics converter.

The first example considers an open-loop unstable non-minimum phase plant, where the generation of stability regions under a fixed delay is validated.

The second example focuses on a DC–DC boost converter. In this case, stability crossing curves are computed, the number of unstable roots is determined through crossing directions, and spectral abscissa optimization is carried out. Finally, a stable open-loop system is considered and stabilized in closed loop using a PI controller that ensures zero steady-state error for step references. Then, by applying Algorithm 1, the spectral abscissa of the closed-loop system is optimized.

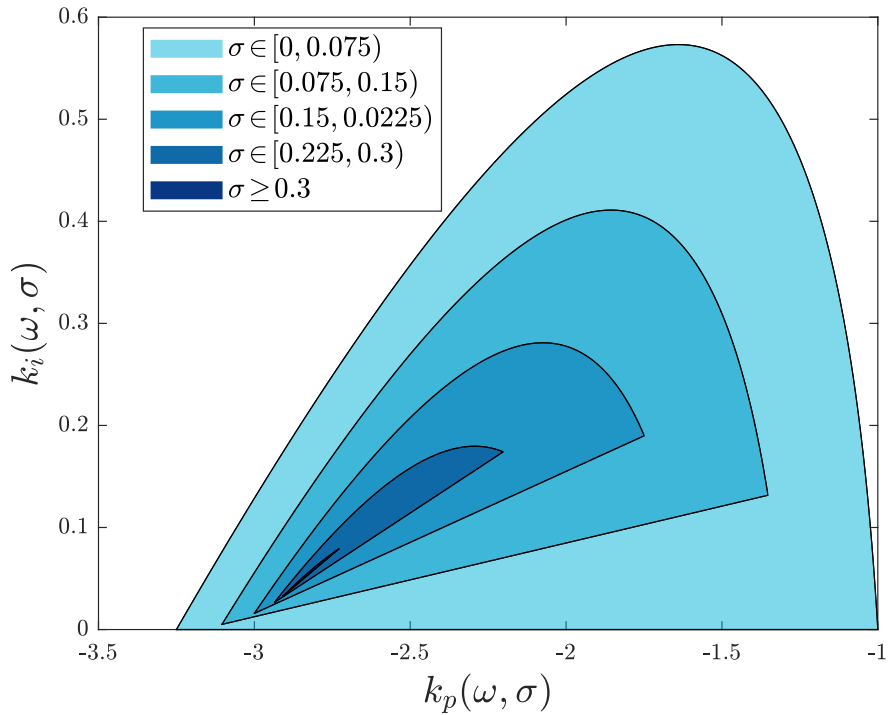
Example 4.6: Unstable open-loop system.

Consider a second-order non-minimum-phase system that is unstable in open loop, described by the transfer function

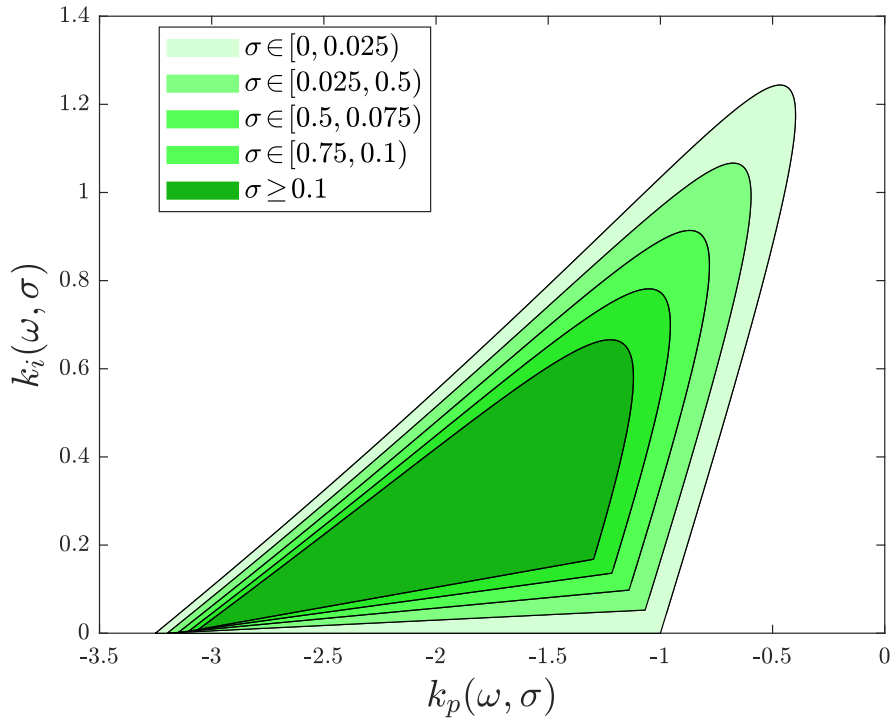
$$G(s) = \frac{1 - \frac{1}{2}s}{s^2 - \frac{1}{2}s + \frac{13}{4}}, \quad (4.73)$$

where $K = \frac{4}{13}$, $\omega_n = \sqrt{\frac{13}{4}}$, $\zeta = -\frac{1}{2\sqrt{13}}$, and $z = \frac{1}{2}$.

This system can be stabilized using a classical PI controller. Furthermore, by applying Propositions 4.14, 4.15, and 4.22, the corresponding σ -stability regions can be computed, as shown in Figure 4.31.



(a) (k_p, k_i) -plane for $\tilde{h} = 0.5$.



(b) (k_p, k_i) -plane for $\tilde{h} = 1$.

Figure 4.31 / σ -stability analysis for unstable system (4.73) for different cases.

From this figure, it can be observed that as the delay h increases, the stability region expands. In addition, the region shifts to the right, thereby avoiding parameter configurations that could introduce unstable zeros into the system.

To further illustrate this behavior with respect to the delay h , Figure 4.32 shows the stability regions in the parameter space (k_p, k_i, h) .

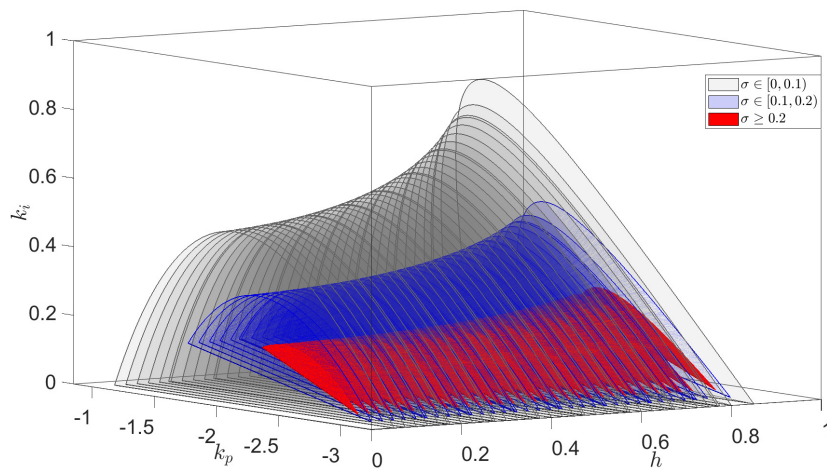


Figure 4.32 / Stability regions for system (4.73).

In particular, it can be observed from Figure 4.32 that the triplet $(k_p, k_i, h) = (-2, \frac{1}{2}, \frac{1}{2})$ stabilizes the system. By applying Proposition 4.26, the corresponding delay margin can be computed.

Indeed, by numerically solving (4.70a)–(4.70c), we obtain $\bar{h} = 2.1616$ s, with the root crossing occurring at $\omega = \frac{1}{2}$. This behavior is illustrated in Figure 4.33.

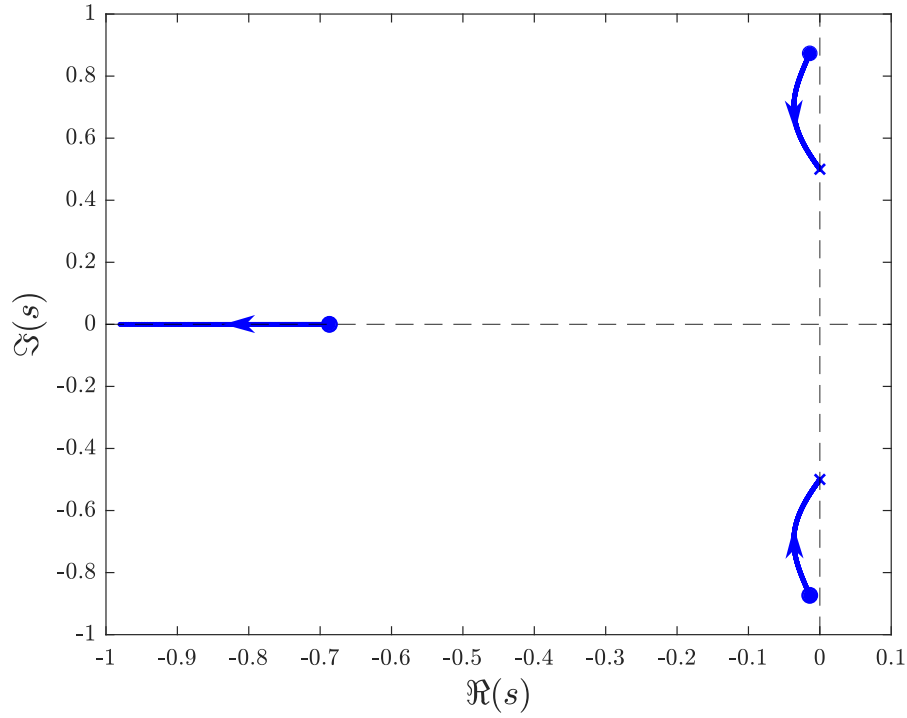


Figure 4.33 / Root locus for example (4.73) with $(k_p, k_i) = (-2, \frac{1}{2})$ and $h \in [0, 2.16]$ s.

Example 4.7: DC–DC Boost Converter

A conventional DC–DC boost converter consists of a voltage source E , a switching device M , an inductor L , a capacitor C , and a load resistor R , as shown in Figure 4.34.

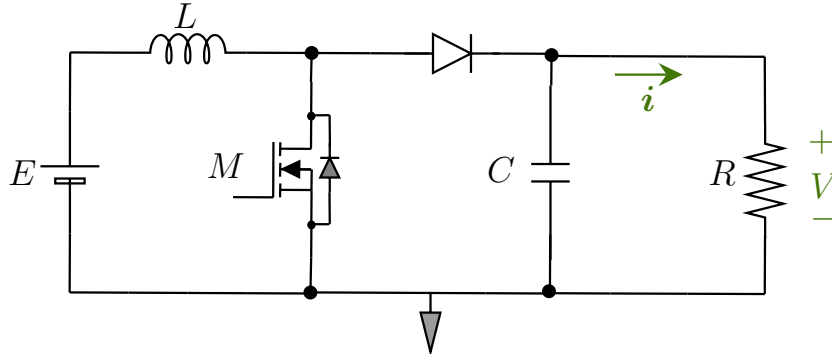


Figure 4.34 / DC-DC Boost Converter.

After straightforward simplifications, the transfer function can be expressed as

$$G(s) = \frac{K\omega_0^2(1 - zs)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \quad (4.74)$$

The system is characterized by a DC gain $K > 0$, a natural frequency $\omega_0 > 0$, an unstable zero located at $s = 1/z > 0$, and a quality factor $Q > 0$, defined as

$$K \triangleq \frac{E}{\eta^2}, \quad \omega_0 \triangleq \frac{\eta}{\sqrt{LC}}, \quad z \triangleq \frac{L}{\eta^2 R}, \quad Q \triangleq \eta R \sqrt{\frac{C}{L}},$$

where $\eta := 1 - D$.

In this example, the parameters are chosen as $E = 12$ V, $V = 24$ V, $D = 0.5$, $C = 10 \mu\text{F}$, $L = 7.05$ mH, and $R = 44 \Omega$. Substituting these values into (4.74) and matching it with the general model (4.39), the resulting parameters are $K = 48$, $\omega_n = 1883.11$, $z = 6.4090 \times 10^{-4}$, and $\zeta = 0.60345$.

Next, by setting $h = 8 \times 10^{-3}$ s and applying Propositions 4.14 and 4.15, the stability crossing curves are obtained, as shown in Figure 4.35.

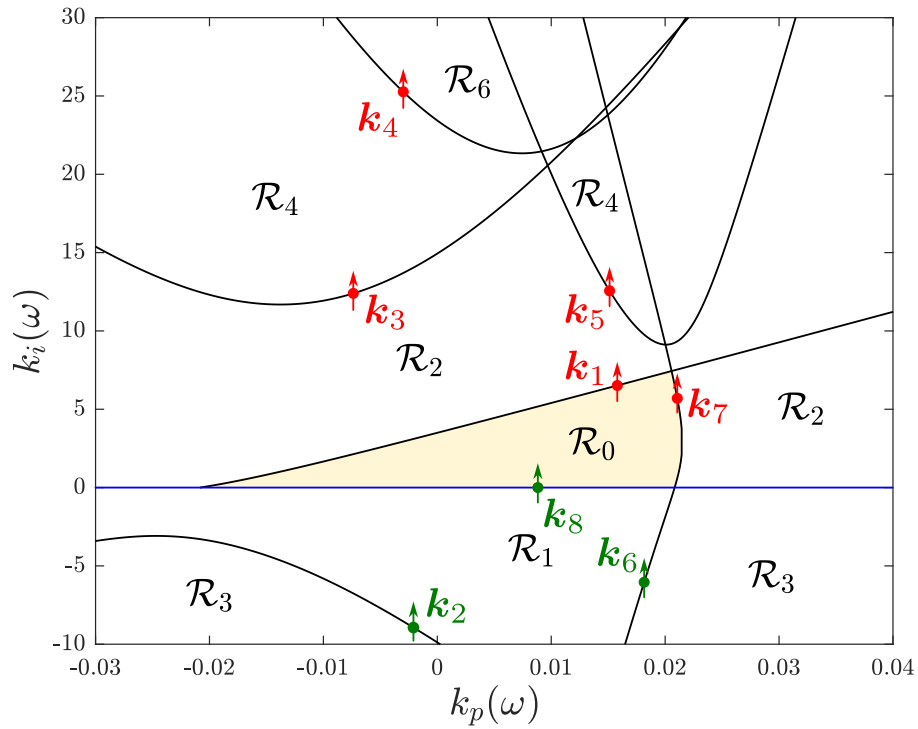


Figure 4.35 / Stability domains and associated crossing directions for (4.74) considering $h = 8 \times 10^{-3}$ s.

Next, to identify the stable regions, the crossing direction defined in Proposition 4.20 is employed. As a first step, consider the set of controllers $\mathbf{k}_j$, with $j \in \{1, 2, \dots, 8\}$, depicted in Figure 4.35 and summarized in Table 4.2.

Controller $\mathbf{k}_j$	Controller gains k_p^* k_i^*		Frequency ω^*	Crossing direction (4.53)
$\mathbf{k}_1$	0.0158	6.5169	180	+
$\mathbf{k}_2$	-0.0021	-8.9360	500	-
$\mathbf{k}_3$	-0.0074	12.3964	800	+
$\mathbf{k}_4$	-0.0030	25.2643	1500	+
$\mathbf{k}_5$	0.0151	12.5617	2300	+
$\mathbf{k}_6$	0.0182	-6.0413	2600	-
$\mathbf{k}_7$	0.0211	5.7005	2800	+
$\mathbf{k}_8$	0.0085	0	0	-

Table 4.2 / Corresponding crossing directions for different controllers in closed-loop with system (4.74).

As mentioned in the Section of Introduction, one of the main challenges in the stability analysis of time-delay systems arises from the infinite, yet countable,

number of characteristic roots. In this context, Proposition 4.20 plays a key role, as it allows one to distinguish between stable and unstable regions.

To illustrate this mechanism, note that at the origin of the (k_p, k_i) -plane, the characteristic function Δ reduces to a polynomial with a finite number of roots. For example, setting $k_i = 0$ and $k_p = 0.0085$ yields the closed-loop characteristic function associated with (4.74), whose roots are given by $\{0, -672.72 \pm i 2130.80\}$.

By applying Proposition 4.20, it can be verified that the region $\mathcal{R}_0$ (see Figure 4.35) is stable, since the root at $s = 0$ crosses toward $\mathbb{C}_-$. Using the same procedure, the remaining regions can be classified as stable or unstable.

A direct application of Proposition 4.21 is to *count* the crossing roots, thereby enabling the identification of stability regions. Figure 4.36 illustrates how five linear trajectories $\overline{\mathcal{O}k_j^\dagger}$, with $j \in \{1, 2, 3, 4, 5\}$, are selected.

Along each trajectory, (4.57) is used to compute the value of η , which determines the number of roots located in the right-half plane (RHP). These calculations are summarized in Table 4.3.

Controller k_j	Controller gains		Frequency		Roots		
	k_p^*	k_i^*	$\text{card}(\Omega_s)$	$I_\varepsilon(\omega_r)$	η_0	$\frac{1 - \text{sign } k_i}{2}$	η
$k_1^\dagger$	0.0346	3.9431	1	$\{1\}$	0	0	2
$k_2^\dagger$	0.0221	27.0335	3	$\{1, 1, 1\}$	0	0	6
$k_3^\dagger$	-0.0070	25.4311	2	$\{1, 1\}$	0	0	4
$k_4^\dagger$	-0.0071	-20.4322	1	$\{1\}$	0	1	3
$k_5^\dagger$	0.0271	-22.5146	2	$\{1, 1\}$	0	1	5

Table 4.3 / Results of the stability analysis algorithm for system (4.74).

The results reported in Table 4.3 are illustrated in Figure 4.36. In this figure, the five selected controllers are depicted, and by applying Proposition 4.21, the corresponding indices used to count the number of roots in the right-half plane (RHP) are obtained.

This approach enables the identification of stability regions without relying on numerical root-finding methods, providing a fast and efficient way to determine the stable region in the parameter plane.

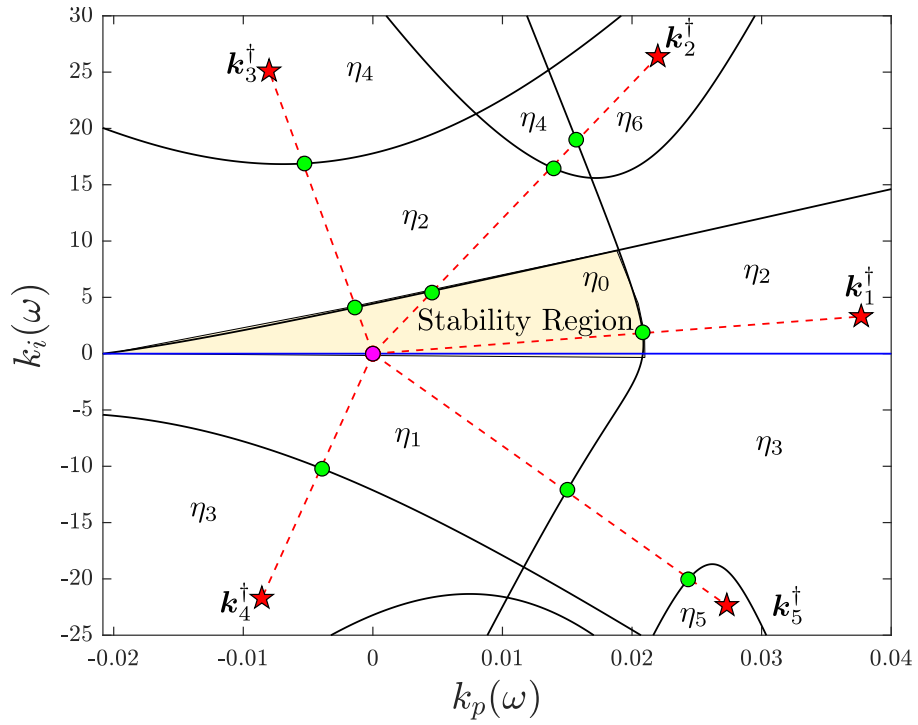


Figure 4.36 / Stability crossing curves and stability regions for system (4.74) with $h = 6 \times 10^{-3}$ s.

Next, to improve the decay rate of the closed-loop system, Proposition 4.22 is considered. Its application is illustrated through two distinct scenarios: (1) $h = 8 \times 10^{-4}$ s and (2) $h = 5 \times 10^{-5}$ s.

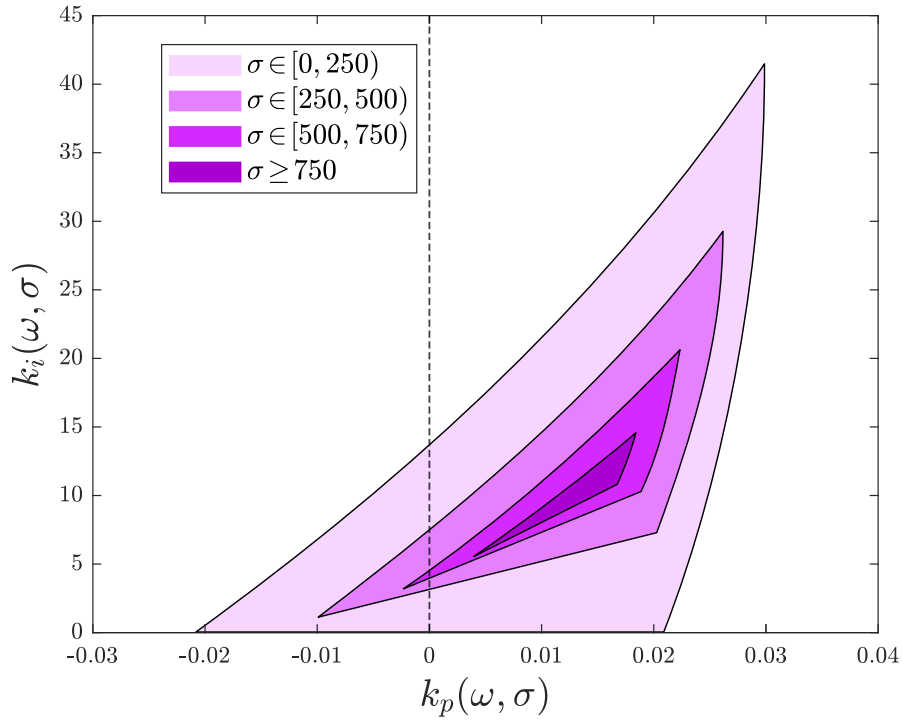


Figure 4.37 / (k_p, k_i) -plane for $h^* = 8 \times 10^{-4}$ s and σ -stability analysis.

Figures 4.37 and 4.38 depict the corresponding stability regions for both scenarios. As observed, the shape of the stability region varies with σ , and in particular, the region shrinks as σ increases.

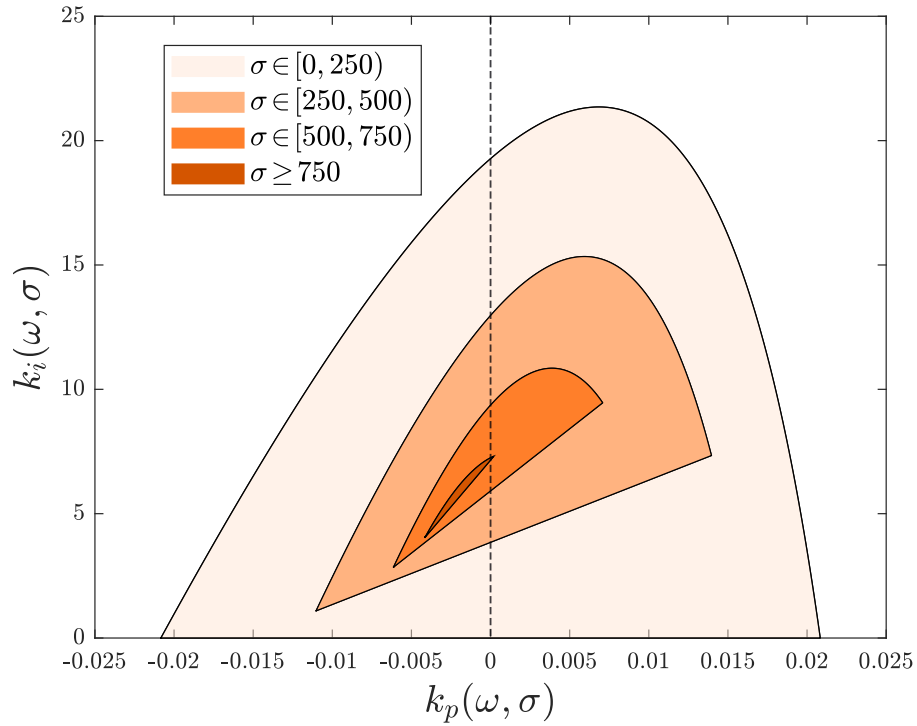


Figure 4.38 / (k_p, k_i) -plane for $h^* = 5 \times 10^{-5}$ s and σ -stability analysis.

A direct consequence of this geometric change is that, under certain conditions, it becomes possible to avoid the introduction of additional unstable zeros while simultaneously enhancing the decay rate by appropriately selecting the delay parameter.

Motivated by the previous observations, the introduction of unstable zeros can be effectively avoided by fixing the parameter k_p and selecting an appropriate delay h such that k_p and k_i share the same sign. This methodology is formalized in Propositions 4.17 and 4.18.

To illustrate this approach, consider $k_p = 0.014$. Then, by applying Proposition 4.24, the corresponding stability region shown in Figure 4.39 is obtained.

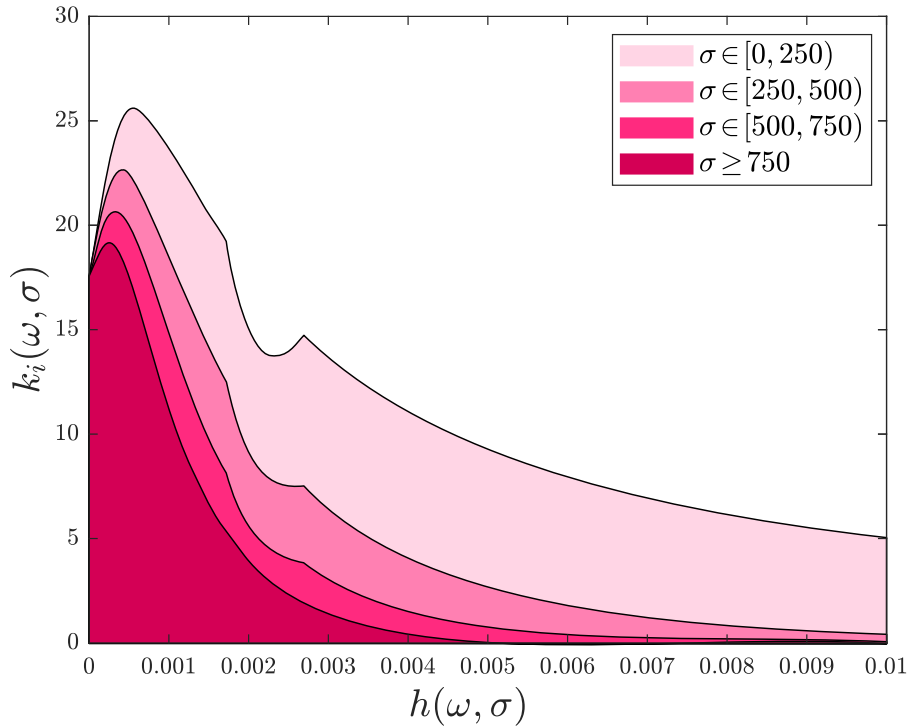


Figure 4.39 / (h, k_i) -plane for $k_p^* = 0.014$ and σ -stability analysis.

Additionally, Table 4.4 reports the spectral abscissa corresponding to various controllers. These results highlight the advantages of employing Propositions 4.22–4.24 for the optimization of the spectral abscissa.

Controller	k_p	k_i	h [s]	α
PI	−0.0039	4.2611	0	−898.10
PδI ₁	−0.002	5.5	0.05×10^{-3}	−781.13
PδI ₂	0.01	8.5	0.08×10^{-3}	−864.31
PδI ₃	0.014	10.4493	0.7567×10^{-3}	−931.72

Table 4.4 / Gains for various controllers in the system (4.74).

Finally, to illustrate the performance of the controllers listed in Table 4.4, Figure 4.40 presents the closed-loop time responses and root locus corresponding to each controller. This figure highlights the influence of the spectral abscissa on the system dynamics.

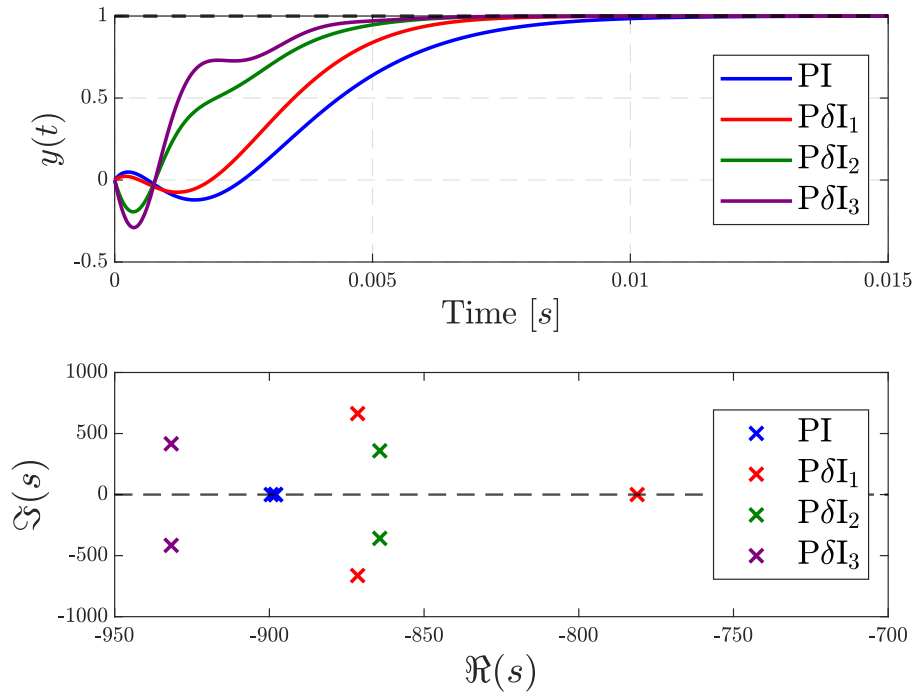


Figure 4.40 / Time step response and root locus of closed loop system formed by (4.74) and different $P\delta I$.

Example 4.8: Stable open-loop system exhibiting non-minimum-phase behavior

Consider the system represented by the following transfer function

$$G(s) = \frac{5.29 - 0.529s}{s^2 + 8.229s + 5.29}, \quad (4.75)$$

where $K = 1$, $\omega_n = \frac{23}{10}$, $\zeta = \frac{17889}{10000}$, and $z = \frac{1}{10}$.

Assume an arbitrary stabilizing PI controller with gains $(k_p, k_i) = (10, 9.4)$. The corresponding closed-loop characteristic roots are located at $s = \{-0.9680, -0.9855 \pm i 7.0953\}$.

Starting from this stable configuration, the objective is to optimize the spectral abscissa by tuning only the delay parameter h . By applying Proposition 4.25 and following the procedure described in Algorithm 2, the results shown in Figure 4.41 are obtained.

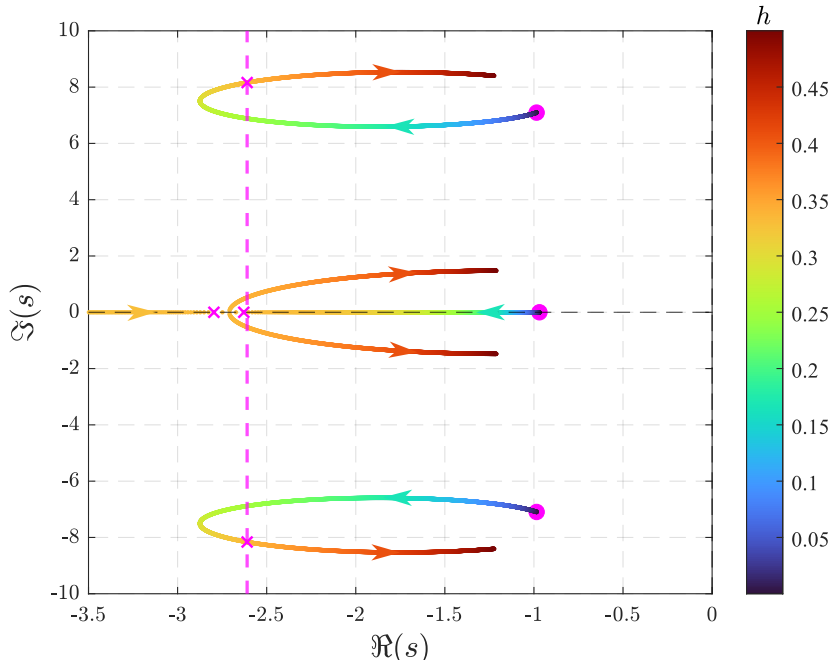


Figure 4.41 / Root locus with PI controller and add the delay value $h \in [0, 0.5]$ s.

Experimental results

The DC–DC boost converter is a power electronic device designed to increase the input voltage to a higher output level while preserving polarity [80]. This type of converter is widely used in applications where the power source operates at a low voltage, such as battery-powered or photovoltaic systems, and a higher voltage level is required to supply the load.

One of the main advantages of DC–DC boost converters is their high power conversion efficiency, as they can achieve efficient operation with minimal losses. This makes them particularly suitable for portable and low-power applications, including mobile devices, digital cameras, and other battery-operated electronics. Moreover, their capability to operate over a wide range of input voltages makes them especially attractive for renewable energy systems.

Despite these advantages, boost converters present several challenges, such as the need for precise control to avoid undesired output fluctuations and the potential generation of electromagnetic interference. To address these issues, advanced control and filtering techniques are typically employed to ensure stable and reliable operation.

In light of the above, the proposed results are validated through both numerical simulations and an experimental implementation based on the DC–DC boost converter introduced in Example 4.7.

Experiment 4.1: DC–DC boost converter

The validation is carried out using the mathematical model described in the numerical section, as well as a laboratory setup illustrated in Figure 4.42.

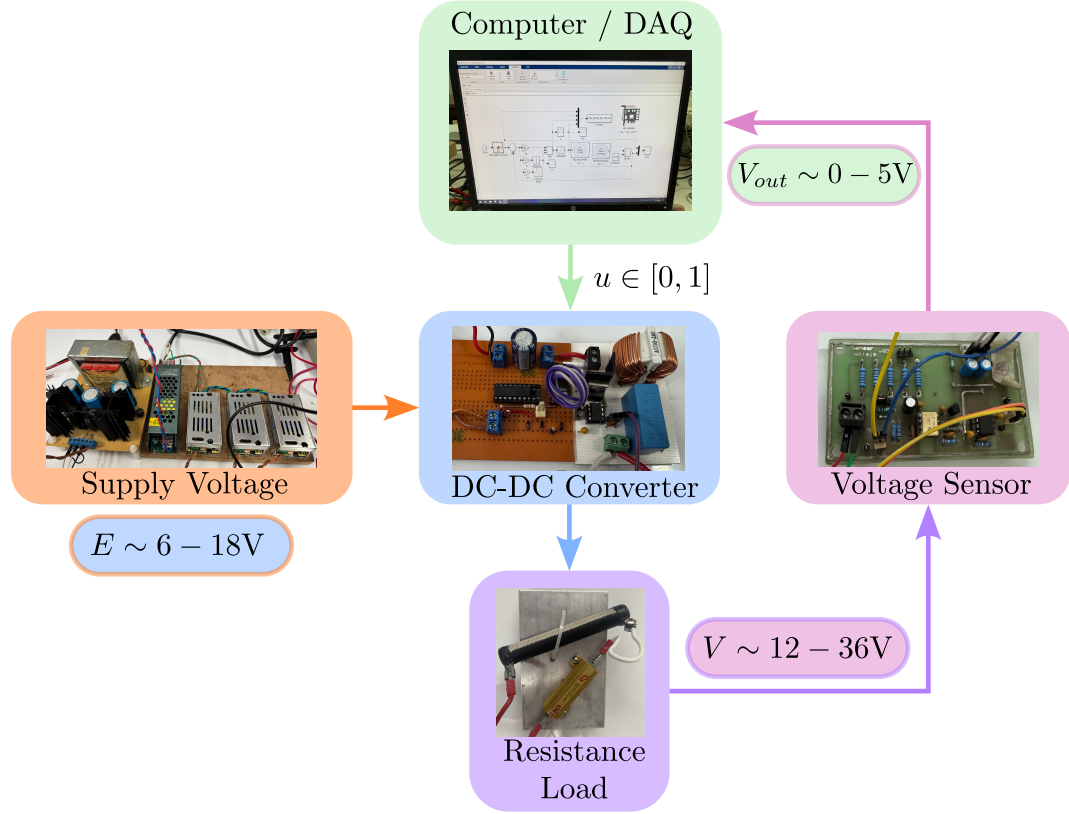


Figure 4.42 / Experimental platform DC–DC converter.

The experimental platform enables the practical implementation and evaluation of the controller performance in a controlled environment. Figure 4.42 consists of five main components: a DC–DC boost converter, a power supply, a load resistor, a voltage sensor that scales down the output voltage (controlled variable), and a data acquisition (DAQ) board.

The scaled voltage is fed back into the DAQ, which transmits the signal to the computer for processing. The control algorithm is executed in real time, and the computed duty cycle (manipulated variable) is sent back through the DAQ to drive the switching element (M) of the converter.

All experiments were carried out in MATLAB/Simulink using the ode1 (Euler) solver for the numerical implementation of the integral action, with a sampling time of $\Delta t = 1$ ms. Each sampling interval is internally divided into 10,000 integration steps to improve numerical accuracy.

The benchmark experiments were performed using a square-wave reference

defined as

$$v_d(t) \triangleq \begin{cases} 24 \text{ V}, & \text{if } 0 < t \leq 2, \\ 12 \text{ V}, & \text{if } 2 < t \leq 4, \\ 24 \text{ V}, & \text{if } t > 4. \end{cases}$$

To enable a fair comparison and assess the previously presented numerical results, three experiments were conducted: one using a stabilizing PI controller (PI_s), another using an optimal PI controller that minimizes the spectral abscissa (PI_σ), and a third using a $\text{P}\delta\text{I}$ controller selected from the numerical examples. The corresponding controller gains are reported in Table 4.5.

Controller	k_p	k_i	h [s]	α
PI_s	0.001	4	0	−241.9
PI_σ	−0.0039	4.2611	0	−898.10
$\text{P}\delta\text{I}$	−0.002	5.5	0.05×10^{-3}	−781.13

Table 4.5 / Gains for the three proposed experiments in the system (4.74).

The three experiments were conducted under identical conditions to ensure a consistent basis for comparison. All tests were performed using the same boost converter setup and a fixed input voltage source of $E = 12$ V, thereby eliminating variability due to hardware or power supply differences.

The resulting responses, shown in Figure 4.43, reflect the effects of the different control strategies under comparable operating conditions.

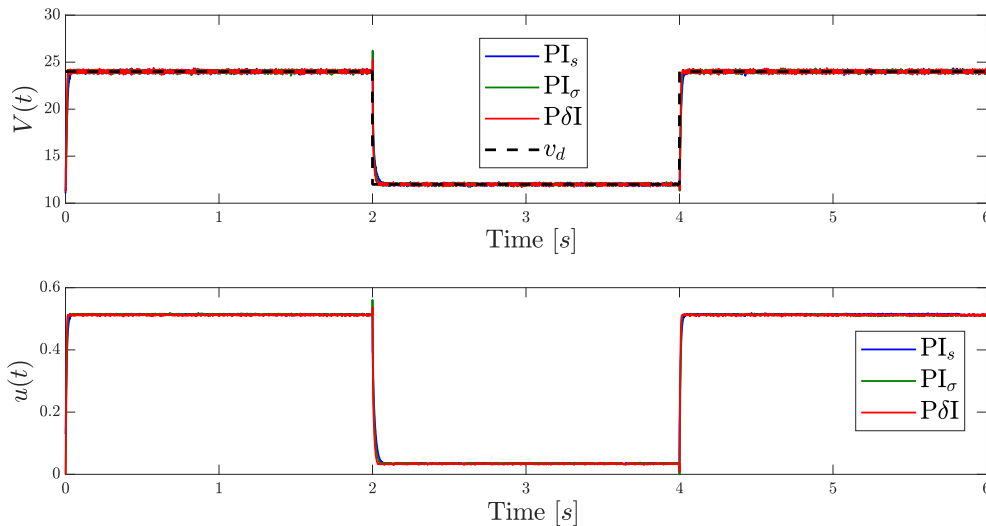


Figure 4.43 / Experimental results of the DC–DC converter.

The previous figure is based on measurements acquired from the DAQ system

and depicts the system response over the entire experiment. As stated throughout this work, the main objective is to enhance the response in terms of the exponential decay of the error to zero.

To further support this analysis, an additional high-resolution measurement was performed using an oscilloscope, focusing on a short time interval within the experiment, specifically $t \in [3.99, 4.06]$. The results of this analysis are presented in Fig. 4.44.

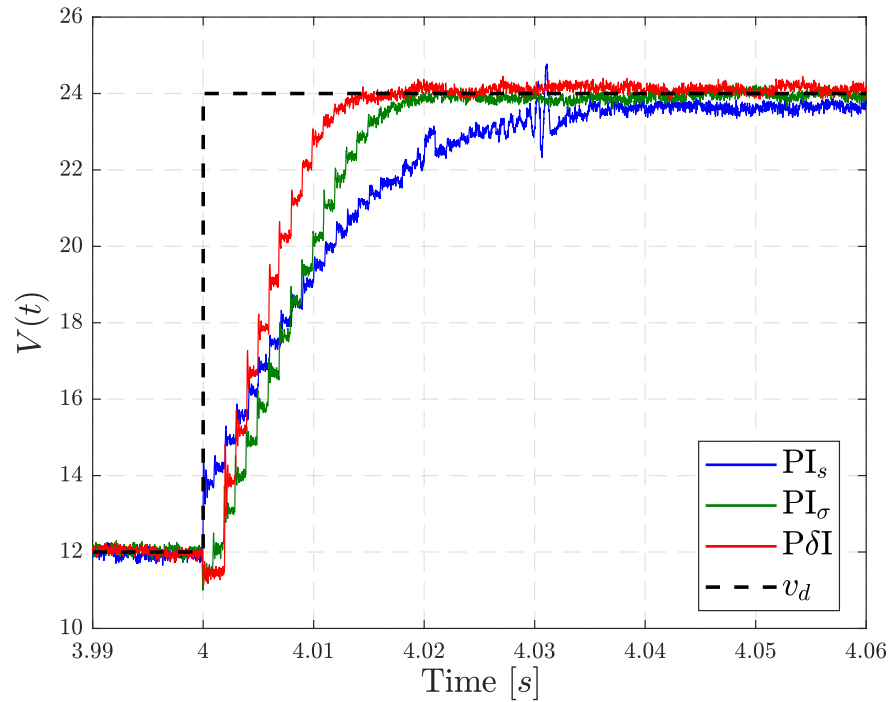


Figure 4.44 / Experimental results of the DC–DC converter in MDO.

From this figure, several previously discussed phenomena can be observed. In particular, the PI_s controller—being the only one for which the gains k_p and k_i share the same sign—does not introduce an unstable zero, which explains why its response does not exhibit undershoot. In contrast, the other two controllers do exhibit this behavior.

Furthermore, in terms of reference tracking, the delay-based controller reaches the desired value the fastest, indicating superior performance in terms of response speed.

4.4 Oscillator

Problem formulation

It is worth noting that the system (4.39) considered in the previous section includes a particular case that requires separate analysis for stability purposes. This case is described below.

Remark 4.12. When $\zeta = z = 0$, the system is referred to as an *even system*. In this case, the transfer function reduces to

$$G(s) \triangleq \frac{K\omega_n^2}{s^2 + \omega_n^2}, \quad (4.76)$$

which satisfies $G(x) = G(-x)$ for all $x \in \mathbb{R}$. ┘

Example 4.9: Oscillator

Let the system dynamics be represented by the transfer function

$$G(s) = \frac{\omega_0^2}{s^2 + \omega_0^2}, \quad \omega_0 \in \mathbb{R}_+. \quad (4.77)$$

It is straightforward to verify that the open-loop system is marginally stable, with poles located at $s_{1,2} = \pm i\omega_0$. Moreover, the characteristic function $\Delta_{2PI}(s; k_p, k_i) : \mathbb{C} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ of the closed-loop system under a classical PI controller is given by

$$\Delta_{PI}(s) = s^3 + \omega_0^2(k_p + 1)s + \omega_0^2k_i. \quad (4.78)$$

A direct application of the Routh–Hurwitz criterion shows that this system cannot be stabilized by a conventional PI controller. In contrast, the use of the PδI controller makes closed-loop stabilization possible. In particular, for $\omega_0 = 1$, the parameter set $(k_p, k_i, h) = (5, 4, 1)$ yields a stable closed-loop response, as shown in Figure 4.45.

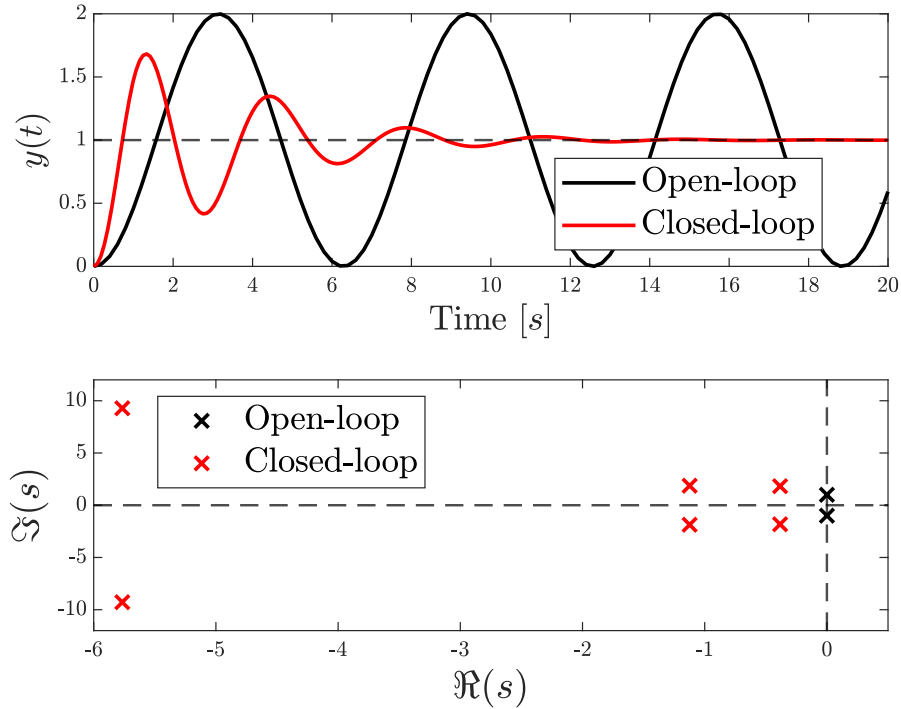


Figure 4.45 / Time responses and root-locus of the system (4.77) for $\omega_0 = 1$.

It is worth noting that a Proportional–Derivative (PD) controller, defined as $C_{PD}(s) \triangleq k_p + k_d s$, can stabilize the system. However, as discussed in the introduction, the derivative action is often undesirable due to its high sensitivity to noise. In practice, this means that rapid variations or measurement noise in the input signal may lead to amplified responses, unwanted oscillations, or even instability.

Another limitation is the reduced robustness in real-world applications. In physical systems, the derivative term is difficult to tune and reliably implement, especially in the presence of time-varying dynamics [81]. Similar issues arise when tuning PID controllers, $C_{PID}(s) \triangleq k_p + \frac{k_i}{s} + k_d s$, which motivates the study of the proposed control strategy and its impact on systems such as the one considered here.

Stability boundaries

PROPOSITION 4.27

Assume that G is an *even* system, and let $h^* > 0$ be fixed. Then, the set of stability crossing curves is given by the following family of lines:

$$k_p \in \mathbb{R}, \quad k_i = 0, \quad (4.79)$$

and

$$k_i = (-1)^\ell \left(\frac{\omega_\ell^3 - \omega_\ell \omega_n^2}{K \omega_n^2} - k_p \omega_\ell \right), \quad k_p \in \mathbb{R}, \quad (4.80)$$

where $\omega_\ell = \frac{(2\ell - 1)\pi}{2h^*}$ and $\ell \in \mathbb{N}$.

Proof. Let ω^* be a singular frequency. From (4.44), it follows that

$$\Delta(i\omega^*; \mathbf{k}, h^*) = 0,$$

which is equivalent to

$$i\omega^* G^{-1}(i\omega^*) + (k_p i\omega^* + k_i e^{-i\omega^* h^*}) = 0.$$

Since $G(s)$ is even, its inverse $G^{-1}(s)$ is also even. Therefore,

$$\Re\{i\omega^* G^{-1}(i\omega^*)\} = 0, \quad \Im\{i\omega^* G^{-1}(i\omega^*)\} = \omega^* G^{-1}(i\omega^*).$$

Taking the real part of $\Delta(i\omega^*; \mathbf{k}, h^*) = 0$ yields

$$k_i \cos(\omega^* h^*) = 0. \quad (4.81)$$

Thus, either $k_i = 0$, which gives (4.79), or $\cos(\omega^* h^*) = 0$, which implies $\omega^* = \omega_\ell$. Substituting this condition into the imaginary part of $\Delta(i\omega^*; \mathbf{k}, h^*) = 0$ yields (4.80), completing the proof. $\square$

Remark 4.13. Although (4.79) coincides with the result in Proposition 4.14, it is included here for completeness. $\lrcorner$

Numerical examples

The example addresses an oscillator-type system, where the proposed results are used to design a stabilizing controller, followed by a robustness analysis with respect to delay variations.

Example 4.10: Oscillator

The study of undamped second-order systems, represented by the transfer function

$$G(s) = \frac{K \omega_n^2}{s^2 + \omega_n^2}, \quad (4.82)$$

is fundamental for understanding oscillatory phenomena, which commonly arise in various fields of science and engineering, such as mechanical structures and electrical circuits.

Consider the parameters $K = 1$ and $\omega_n = 1$. As shown in the motivating example (see Example 4.9), this class of systems cannot be stabilized using a classical PI controller. However, replacing the PI controller with a P δ I controller enables stabilization.

Specifically, the gains $(k_p, k_i) = (5, 4)$ are selected, and the closed-loop response and spectrum are analyzed as the delay h varies. The corresponding results are illustrated in Figure 4.46.

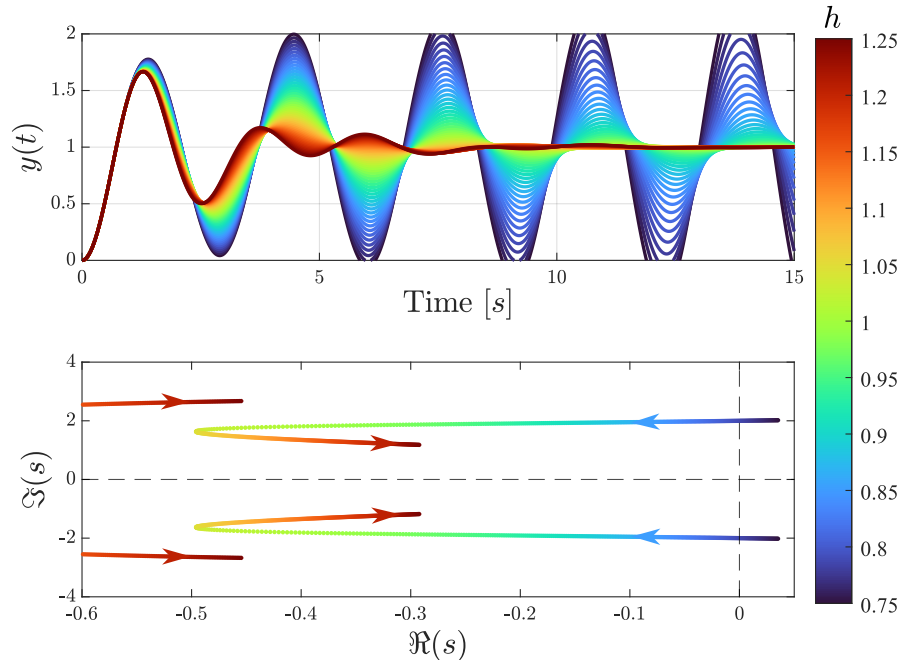
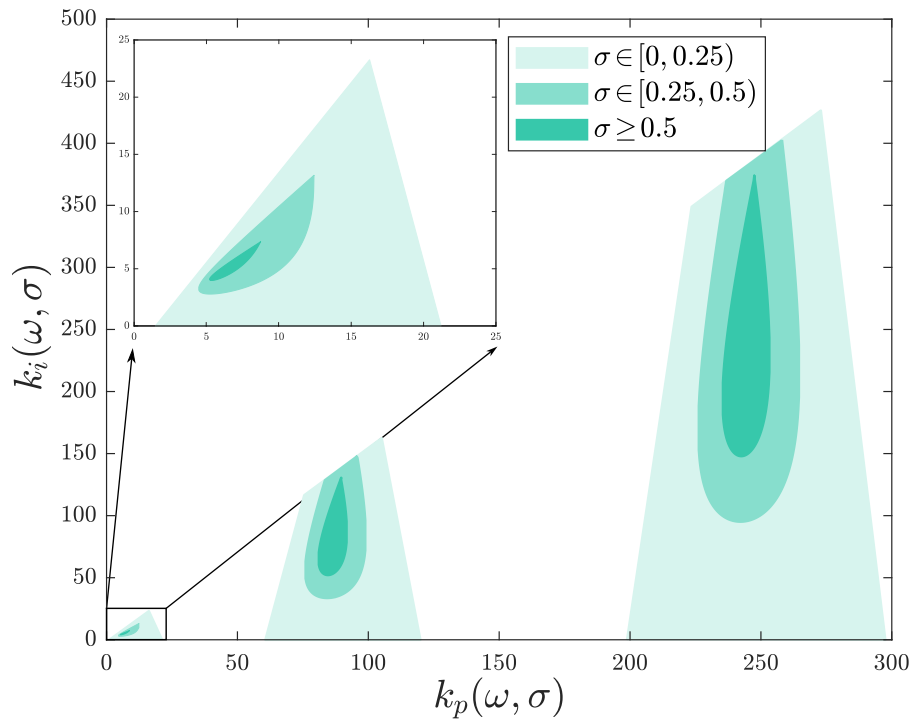


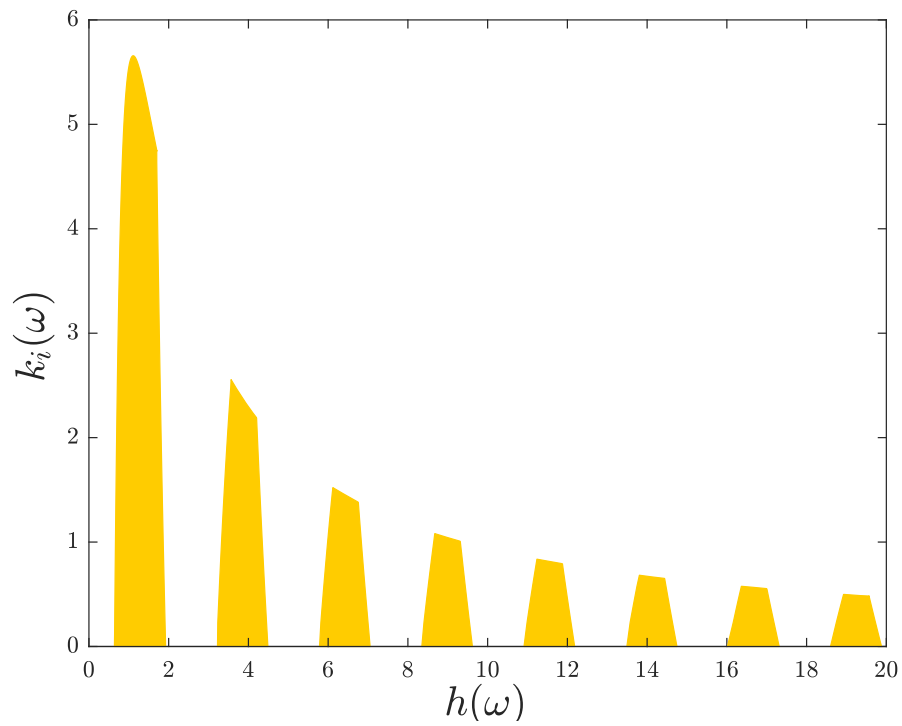
Figure 4.46 / Root locus and time step response P δ I with $h \in [0.75, 1.25]$ s.

In the following, we assume $h = 1$ s. By applying Proposition 4.27, the corresponding stability region is obtained, as shown in Figure 4.47a. This figure also includes the σ -regions derived from Proposition 4.22.

Similarly, for a fixed value $\tilde{k}_p = 5$, Figure 4.47b shows the stability regions obtained from Propositions 4.17 and 4.18.



(a) (k_p, k_i) -plane for $h = 1$ and several σ .



(b) (h, k_i) -plane for $k_p^* = 5$.

Figure 4.48 / Stability region analysis for the even system (4.82).

For comparison, three controllers with fixed delays are selected from distinct regions identified in the analysis. The corresponding controller gains and spectral abscissa values of the closed-loop system are summarized in Table 4.6.

Controller	k_p	k_i	h [s]	α
$P\delta I_1$	6.2	4.5	1	-0.5611
$P\delta I_2$	86	60	1	-0.5800
$P\delta I_3$	245	160	1	-0.5718

Table 4.6 / Different stabilizing controller gains for system (4.82).

Although the controllers are chosen from three different regions (see Figure 4.47a), each characterized by significantly different gain magnitudes, the resulting spectral abscissa of the closed-loop system is nearly identical in all cases.

Finally, the closed-loop response of the system is evaluated for the three proposed controllers, showing a similar decay rate in all cases. However, it is observed that larger control gains lead to higher oscillation frequencies, which is often undesirable from a performance perspective.

This behavior is illustrated in Figure 4.49, where both the closed-loop poles and the step response of the system are presented.

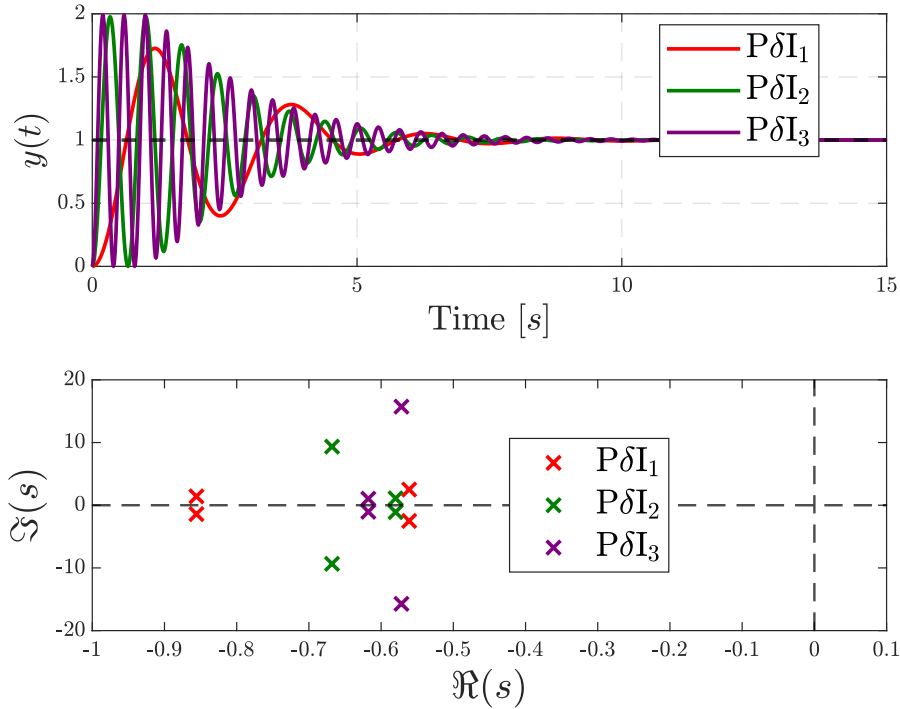


Figure 4.49 / Time step response and root locus of closed loop system formed by (4.82) and different $P\delta I$.

4.5 Double Integrator

Double integrator systems arise naturally in a wide range of physical applications where the control input directly affects the acceleration of the system, and the output corresponds to position.

This structure appears, for instance, in translational mechanical systems without friction, where the dynamics are governed by Newton's second law $F = m\ddot{x}$, leading to a transfer function of the form $X(s)/F(s) = 1/(ms^2)$. Similarly, in rotational systems with negligible damping, the equation $\tau = J\ddot{\theta}$ results in $\Theta(s)/T(s) = 1/(Js^2)$.

Moreover, simplified models of vehicles, robotic manipulators, and aerial systems often exhibit double integrator behavior when dissipative effects are neglected, yielding dynamics of the form $\ddot{x} = u$. These models are widely used in control design due to their simplicity and their ability to capture the dominant inertial behavior of the system.

In general, a double integrator represents a purely inertial system with no inherent damping, where the input influences acceleration, which is then integrated twice to produce position. This lack of dissipation makes such systems particularly challenging from a control perspective, as they require appropriate feedback strategies to ensure stability and desired performance.

Problem formulation

Consider the plant described by the transfer function

$$G(s) = \frac{1}{s^2}, \quad (4.83)$$

which corresponds to a double integrator system. As is well known, such systems lack inherent damping and require feedback control to ensure stability and acceptable transient performance.

A natural choice for controlling this type of system is a proportional-derivative (PD) controller, since the derivative action introduces damping and allows direct shaping of the closed-loop dynamics. However, as discussed in the Introduction, the use of the derivative term is often undesirable in practical implementations due to its high sensitivity to measurement noise, which may lead to signal amplification and degraded performance.

Motivated by this limitation, an alternative approach based on delay is considered. In particular, a proportional delayed integral (P δ I) controller is adopted, whose structure replaces the derivative action by a delayed signal. This strategy preserves the capability of modifying the system dynamics while avoiding the noise amplification issues associated with the derivative term.

Therefore, the P δ I controller provides a suitable compromise between performance and robustness, making it an attractive solution for the control of double integrator systems.

In order to study its stability, the characteristic equation of the closed-loop system rewrites as $C(s)G(s) + 1 = 0$, which leads to:

$$\Delta(s; k_p, k_i, h) := s^3 + k_p s + k_i e^{-hs}. \quad (4.84)$$

Analytical approach

To derive the stability conditions in terms of the parameters (k_p, k_i, h) , we first analyze the system for $\tau = 0$. In this case, the quasi-polynomial reduces to a standard algebraic polynomial, whose roots are obtained from

$$s^3 + k_p s + k_i = 0.$$

It can be readily verified that this polynomial does not satisfy the Hurwitz stability conditions. Therefore, the introduction of a strictly positive delay becomes necessary, leading to the first stability requirement

$$h > 0. \quad (4.85)$$

Based on this observation, and considering a fixed positive delay, the following conditions are derived to tune the gains k_p and k_i in order to guarantee closed-loop

stability.

$$\left(\frac{(2\ell-1)\pi}{2\tau}\right)^2 < k_p < \left(\frac{(2\ell+1)\pi}{2\tau}\right)^2, \quad (4.86)$$

$$0 < k_i < k_p \left(\frac{\pi}{2\tau}\right) - \left(\frac{\pi}{2\tau}\right)^3. \quad (4.87)$$

where $\ell \in \mathbb{N}$.

Stability crossing curves

One approach to tune the parameters (k_p, k_i, h) is through the use of stability maps, which are defined as follows.

PROPOSITION 4.28

(CRB-RRB) Let $h > 0$ be a fixed value. Then, the set of all stability crossing curves is given by the following set of lines:

$$k_i = 0, \quad k_p \in \mathbb{R}, \quad (4.88)$$

$$k_i = (-1)^\ell (\omega_\ell^3 - k_p \omega_\ell), \quad k_p \in \mathbb{R}, \quad (4.89)$$

where $\omega_\ell = \frac{(2\ell-1)\pi}{2h}$ and $\ell \in \mathbb{N}$.

Proof. Substituting $s = i\omega$ in (4.84) we obtain the following

$$-i\omega^3 + i\omega k_p + k_i(\cos(\omega h) - i \sin(\omega h)), \quad (4.90)$$

the real part and imaginary part of (4.90), obtain

$$\begin{cases} k_i \cos(\omega h) = 0, \\ -\omega^3 + k_p \omega - k_i \sin(\omega h) = 0. \end{cases} \quad (4.91)$$

The first equation of the previous case is zero when $k_i = 0$ or $\cos(\omega h) = 0$, the second case is fulfilled when ω is an odd integer multiple of $\frac{\pi}{2h}$ therefore, two equations are obtained. The one case as

$$\begin{cases} k_i = -\omega_\ell^3 + k_p \omega_\ell, & k_p \in \mathbb{R}, \\ \omega_\ell = \frac{\pi}{2h}, \frac{5\pi}{2h}, \dots \end{cases} \quad (4.92)$$

The other case is defined as

$$\begin{cases} k_i = \omega_\ell^3 - k_p \omega_\ell, & k_p \in \mathbb{R}, \\ \omega_\ell = \frac{3\pi}{2h}, \frac{7\pi}{2h}, \dots \end{cases} \quad (4.93)$$

Therefore, the lines (4.92)-(4.93) can be rewritten as follows:

$$\begin{cases} k_i = (-1)^\ell (\omega_\ell^3 - k_p \omega_\ell), \\ \omega_\ell = \frac{(2\ell-1)\pi}{2h}, \quad \ell \in \mathbb{N}. \end{cases} \quad (4.94)$$

□

PROPOSITION
4.29

Let $h > 0$ be a fixed value, $\sigma > 0$, the set of all stability crossing curves is composed by

$$k_p(\omega, \sigma) = 3\omega^2 - \sigma^2 + \frac{2\omega(\omega^2 + \sigma^2) \cos(\omega h)}{\sigma \sin(\omega h) - \omega \cos(\omega h)} \quad (4.95a)$$

$$k_i(\omega, \sigma) = e^{-h\sigma} \frac{2\omega(\omega^2 + \sigma^2)\sigma}{\sigma \sin(\omega h) - \omega \cos(\omega h)}. \quad (4.95b)$$

Proof. The proof follows the same procedure as in Propositions 4.14–4.15, under the change of variable $s = -\sigma + i\omega$. $\square$

Numerical examples

Example 4.11: Double Integrator

Consider the system described by the transfer function

$$G(s) = \frac{1}{s^2}, \quad (4.96)$$

To illustrate the proposed methodology, consider the system (4.96) under a PδI control scheme. The stability regions are obtained by analyzing the roots of the corresponding quasi-polynomial and determining the set of controller parameters for which all roots lie in the open left-half complex plane.

This is achieved by computing the stability crossing boundaries through the substitution $s = i\omega$, which allows the parametrization of the controller gains as functions of the crossing frequency. Based on this procedure, the stability regions can be constructed in the (k_p, k_i) -plane for fixed values of the delay parameter.

Figure 4.50 shows the lines defined by the values of ℓ , which generate polygonal regions corresponding to sets of stabilizing gains. This representation is two-dimensional and corresponds to a fixed delay value. In this example, the delay is set to $h = 0.5$ s.

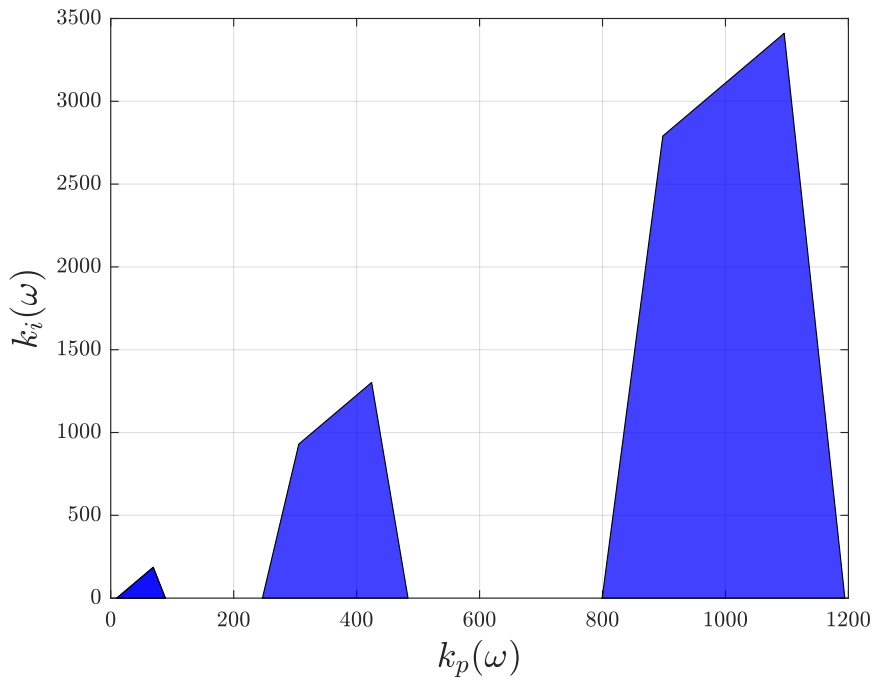


Figure 4.50 / Stability regions (k_p, k_i) where $\ell = 1, 2, \dots, 6$ for a h fixed.

Figure 4.51 illustrates the evolution of the delay-based controller parameters as h varies from 0.5 s to 1.5 s. In particular, it can be observed that the stability regions shrink as the delay increases, thereby reducing the range of admissible controller gains.

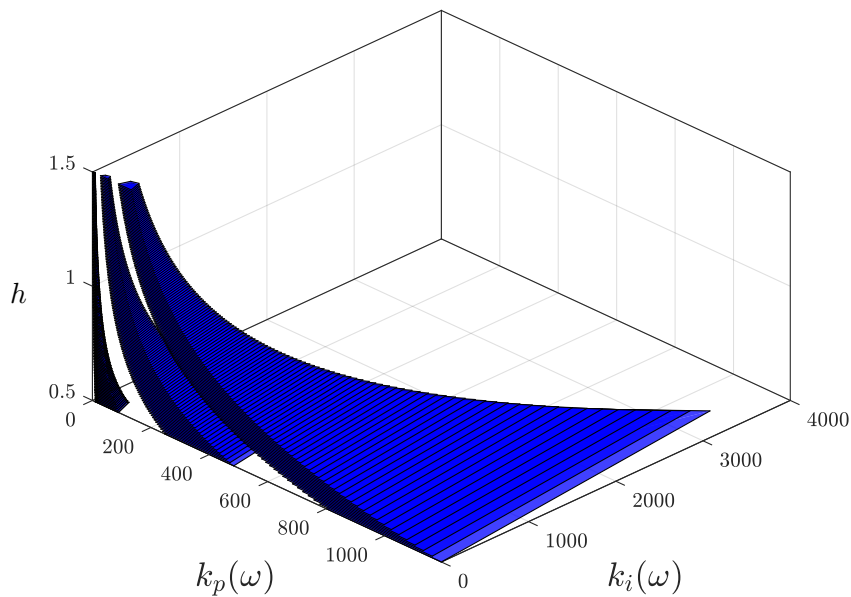


Figure 4.51 / Stability regions (k_p, k_i, h) where $\ell = 1, 2, \dots, 6$.

In this case, the analysis is extended by shifting the imaginary axis to $s = -\sigma + i\omega$, ensuring that all closed-loop poles lie to the left of the vertical line $\text{Re}(s) = -\sigma$. This provides a quantitative measure of the exponential decay rate of the system response and enables the selection of controller parameters that guarantee a desired level of performance.

For a fixed delay value h , the stability regions are delimited by the stability crossing curves shown in Figure 4.52. Within these boundaries, a range of admissible combinations of (k_p, k_i) is identified, as highlighted by the yellow region, which guarantees closed-loop stability.

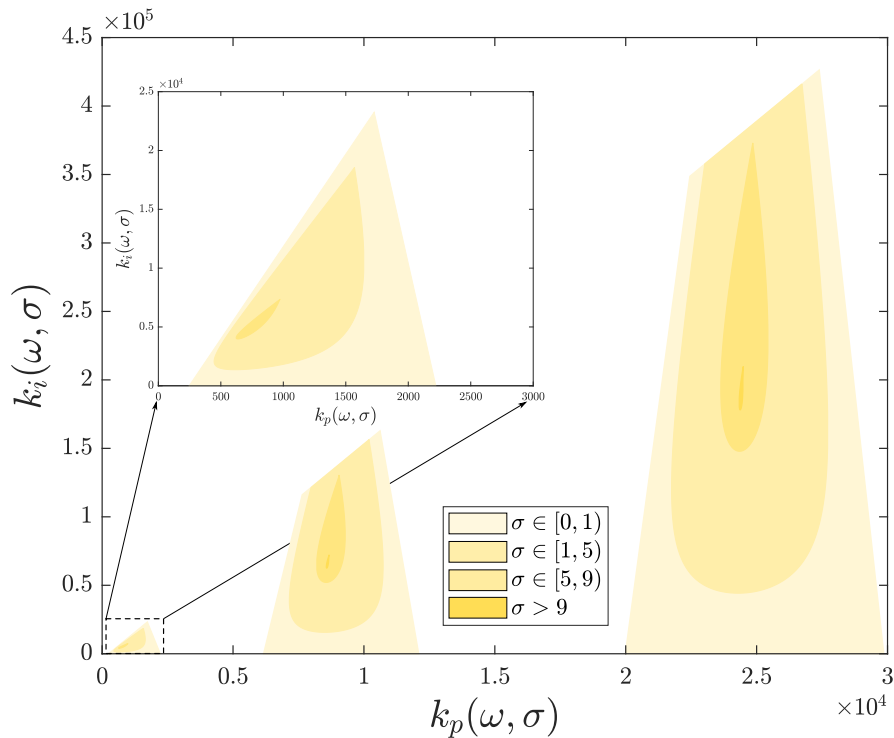


Figure 4.52 / Stability regions (k_p, k_i) where $0 < \sigma < 9$ and $\tau = 0.1$.

4.6 Delay-Based State Feedback Control for Inverted Pendulum

An inertial inverted pendulum is a classical nonlinear dynamical system consisting of a rigid body that must be balanced in an unstable upright position by means of an external control input, typically torque or force. Its dynamics are inherently unstable and highly sensitive to disturbances, which makes it a benchmark problem in control engineering.

From a control perspective, the inverted pendulum captures fundamental challenges such as stabilization of unstable equilibria, fast transient response, and robustness against uncertainties. Due to its simplicity and rich dynamics, it has

been extensively used to validate control strategies, ranging from classical linear controllers to advanced nonlinear and delay-based approaches, making it a key testbed for both theoretical developments and experimental implementations.

System modeling

The model derivation is based on the diagram shown in Figure 4.53, where the angle q_p is defined such that $q_p = 0$ corresponds to the unstable equilibrium position of the pendulum. This convention facilitates subsequent system implementation and linearization.

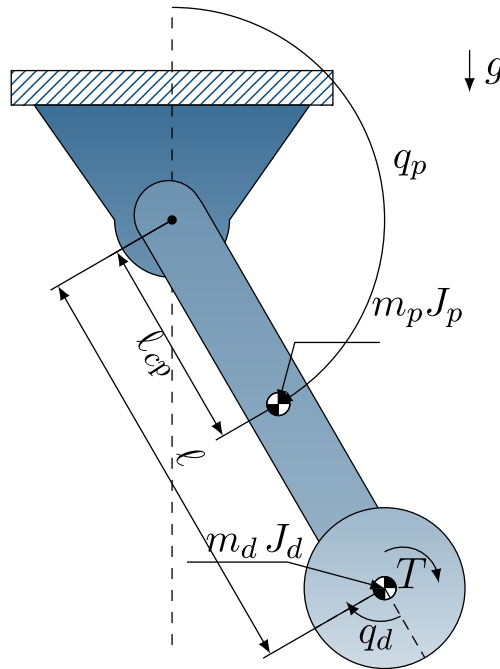


Figure 4.53 / Schematic representation of the system.

From Figure 4.53, it can be observed that g denotes the gravitational acceleration. The masses are represented by m , and the inertias by J , while the subscripts p and d refer to the pendulum and the disk, respectively.

The parameter ℓ corresponds to the distance from the disk's center of gravity to the origin, whereas ℓ_{cp} denotes the distance from the pendulum's center of gravity to the origin. Additionally, q_p represents the angular position of the pendulum with respect to the origin (first generalized coordinate), and q_d denotes the angular position of the disk relative to the pendulum (second generalized coordinate). Finally, T corresponds to the generalized force applied to the system.

Finally, in the modeling process, friction effects between the pendulum and its pivot, the disk and the motor, as well as those arising from shaft misalignment, are neglected. Therefore, by applying the Euler–Lagrange formulation, the following

model is obtained:

$$\begin{bmatrix} J_p + m_p \ell_{cp}^2 + m_d \ell^2 + J_d & J_d \\ J_d & J_d \end{bmatrix} \begin{bmatrix} \ddot{q}_p \\ \ddot{q}_d \end{bmatrix} - \begin{bmatrix} (m_p \ell_{cp} + m_d \ell) g \sin(q_p) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix}, \quad (4.97)$$

for further details, see [82] [83], and references therein.

Based on the previous model, several observations can be made.

- The system is nonlinear; however, it can be linearized around values of q_p close to zero. This allows assessing its stabilizability at the unstable equilibrium point, as well as determining the type of stability it exhibits.
- The state-space representation of the linearized system requires at least three state variables. If the system is shown to be controllable, then it can be stabilized through full state feedback.
- Since the system is conservative, its dynamics can be analyzed in terms of energy to develop nonlinear control strategies.
- The system is autonomous, meaning that it does not explicitly depend on time. This implies the existence of an underlying symmetry, and therefore the Hamiltonian coincides with the total energy of the system. Moreover, since the system does not depend on q_d , this variable is cyclic, which allows deriving an associated conservation law.

At the beginning of the control design, the first and most important step is to eliminate the nonlinear term $\sin(q_p)$ from the equation of motion of q_p (see equation (4.97)). A common approach is to approximate the sine function using its Taylor series expansion around 0 [77], given by

$$\sin(q_p) = \sum_{n=0}^{\infty} (-1)^n \frac{q_p^{2n+1}}{(2n+1)!}. \quad (4.98)$$

If q_p is sufficiently small, the following approximation can be used:

$$\sin(q_p) \approx q_p.$$

It can be shown that this linearization is valid within the range ± 0.1818 rad (approximately $\pm 10.417^\circ$), with an error bound given by $|q_p - \sin(q_p)| < 1 \times 10^{-3}$.

Although this approach allows for a straightforward linearization of the model, a state-space linearization technique is presented next, as it will be required in subsequent developments.

First, the system described by equation (4.97) is expressed in matrix form as follows:

$$\mathbf{D}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u}(t) \quad (4.99)$$

The first observation from equation (4.97) is that the matrix $\mathbf{D}$ is invertible, since

$$\det(\mathbf{D}) = \begin{vmatrix} m_p \ell_{cp}^2 + m_d \ell^2 + J_p + J_d & J_d \\ J_d & J_d \end{vmatrix} = (m_p \ell_{cp}^2 + m_d \ell^2 + J_p + J_d) J_d - J_d^2,$$

which simplifies to

$$\det(\mathbf{D}) = (m_p \ell_{cp}^2 + m_d \ell^2 + J_p) J_d - J_d^2 \neq 0.$$

Therefore, equation (4.97) can be solved for $\ddot{\mathbf{q}}$ as

$$\ddot{\mathbf{q}} = \mathbf{D}(\mathbf{q})^{-1} [\mathbf{u}(t) - \mathbf{g}(\mathbf{q})]. \quad (4.100)$$

Expanding this expression, we obtain

$$\begin{bmatrix} \ddot{q}_p \\ \ddot{q}_d \end{bmatrix} = \frac{1}{\det(\mathbf{D})} \begin{bmatrix} D_{22} & -D_{21} \\ -D_{12} & D_{11} \end{bmatrix} \begin{bmatrix} (m_p \ell_{cp} + m_d \ell) g \sin(q_p) \\ T(t) \end{bmatrix}. \quad (4.101)$$

From this, the equations of motion can be written as

$$\ddot{q}_p = \frac{D_{22}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin(q_p) - \frac{D_{21}}{\det(\mathbf{D})} T(t), \quad (4.102)$$

$$\ddot{q}_d = -\frac{D_{12}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin(q_p) + \frac{D_{11}}{\det(\mathbf{D})} T(t). \quad (4.103)$$

Next, we define the state vector as

$$\mathbf{x} = [x_1, x_2, x_3]^\top = [q_p, \dot{q}_p, \dot{q}_d]^\top.$$

Through this state vector, the system can be expressed in state-space form. The system can be expressed in state-space form through the state vector as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}) = [f_1(\mathbf{x}^\top, \mathbf{u}^\top), f_2(\mathbf{x}^\top, \mathbf{u}^\top), f_3(\mathbf{x}^\top, \mathbf{u}^\top)]^\top$ is a vector of nonlinear functions, and $\mathbf{u} = [u_1, u_2, u_3]^\top$ is the input vector. In this case, there is a single input, given by $\mathbf{u} = [T(t)]^\top$.

By rewriting equations (4.102) and (4.103) in terms of the state vector, we obtain

$$\dot{x}_2 = \frac{D_{22}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin(x_1) - \frac{D_{21}}{\det(\mathbf{D})} T(t), \quad (4.104)$$

$$\dot{x}_3 = -\frac{D_{12}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin(x_1) + \frac{D_{11}}{\det(\mathbf{D})} T(t). \quad (4.105)$$

The nonlinear state-space model can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}^T, T(t)) \\ f_2(\mathbf{x}^T, T(t)) \\ f_3(\mathbf{x}^T, T(t)) \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{D_{22}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin x_1 - \frac{D_{21}}{\det(\mathbf{D})} T(t) \\ -\frac{D_{12}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin x_1 + \frac{D_{11}}{\det(\mathbf{D})} T(t) \end{bmatrix}. \quad (4.106)$$

To linearize the system, we seek equilibrium points $(\hat{\mathbf{x}}, \hat{T})$ such that

$$\mathbf{f}(\hat{\mathbf{x}}, \hat{T}) = 0,$$

that is,

$$\begin{bmatrix} \hat{x}_2 \\ \frac{D_{22}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin \hat{x}_1 - \frac{D_{21}}{\det(\mathbf{D})} \hat{T} \\ -\frac{D_{12}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin \hat{x}_1 + \frac{D_{11}}{\det(\mathbf{D})} \hat{T} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (4.107)$$

From the first equation, it follows directly that

$$\hat{x}_2 = 0. \quad (4.108)$$

From the second and third equations, we obtain

$$\begin{aligned} \frac{D_{22}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin \hat{x}_1 - \frac{D_{21}}{\det(\mathbf{D})} \hat{T} &= 0, \\ -\frac{D_{12}}{\det(\mathbf{D})} (m_p \ell_{cp} + m_d \ell) g \sin \hat{x}_1 + \frac{D_{11}}{\det(\mathbf{D})} \hat{T} &= 0. \end{aligned}$$

Solving for $\hat{T}$ from both expressions yields

$$\hat{T} = \frac{D_{22}}{D_{21}} (m_p \ell_{cp} + m_d \ell) g \sin(\hat{x}_1), \quad (4.109)$$

$$\hat{T} = \frac{D_{12}}{D_{11}} (m_p \ell_{cp} + m_d \ell) g \sin(\hat{x}_1). \quad (4.110)$$

By equating expressions (4.109) and (4.110), we obtain

$$\left(\frac{D_{22}}{D_{21}} (m_p \ell_{cp} + m_d \ell) g - \frac{D_{12}}{D_{11}} (m_p \ell_{cp} + m_d \ell) g \right) \sin(\hat{x}_1) = 0. \quad (4.111)$$

For this expression to hold, at least one of the factors must be zero. By direct algebraic manipulation, the first term is proportional to

$$D_{22}D_{11} - D_{12}D_{21} = \det(\mathbf{D}),$$

which has already been shown to be nonzero. Therefore, the only possible solution is

$$\sin(\hat{x}_1) = 0,$$

which implies

$$\hat{x}_1 = \pm 2k\pi, \quad k = 0, 1, 2, \dots \quad (4.112)$$

Substituting this result (4.112) into either (4.109) or (4.110), it follows that

$$\hat{\tau} = 0. \quad (4.113)$$

Finally, a condition for $\hat{x}_3$ can be obtained by substituting (4.108), (4.112), and (4.113) into equation (4.105), yielding

$$\hat{x}_3 = -\frac{D_{12}}{\det(D)} (m_p \ell_{cp} + m_d \ell) g \sin(\hat{x}_1) + \frac{D_{11}}{\det(D)} \hat{T} = 0. \quad (4.114)$$

$$\hat{x}_3 = c, \quad (4.115)$$

where c is a constant. This expression not only provides a constraint but also offers insight into the system behavior: when the pendulum is at $q_p = 0$, the disk velocity must remain constant.

The system can be written in state-space form as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\boldsymbol{\omega}, \quad (4.116)$$

where

$$\mathbf{z} = \mathbf{x} - \hat{\mathbf{x}}, \quad \boldsymbol{\omega} = T - \hat{T} = T, \quad (4.117)$$

and the matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right|_{(\hat{\mathbf{x}}, \hat{T})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}_{(\hat{\mathbf{x}}, \hat{T})}, \quad (4.118)$$

$$\mathbf{B} = \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{(\hat{\mathbf{x}}, \hat{T})} = \begin{bmatrix} \frac{\partial f_1}{\partial T} \\ \frac{\partial f_2}{\partial T} \\ \frac{\partial f_3}{\partial T} \end{bmatrix}_{(\hat{\mathbf{x}}, \hat{T})}. \quad (4.119)$$

By applying the state-space representation (4.118)–(4.119) and introducing the notation

$$\frac{D_{ij}}{\det(\mathbf{D})} = \bar{D}_{ij}, \quad (m_p \ell_{cp} + m_d \ell) = \bar{m},$$

to simplify expressions, the matrices $\mathbf{A}$ and $\mathbf{B}$ evaluated at the equilibrium point $(\hat{\mathbf{x}}, \hat{T})$ are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{D}_{22} \bar{m} g \cos x_1 & 0 & 0 \\ -\bar{D}_{12} \bar{m} g \cos x_1 & 0 & 0 \end{bmatrix}_{(\hat{x}, \hat{\tau})} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{D}_{22} \bar{m} g & 0 & 0 \\ -\bar{D}_{12} \bar{m} g & 0 & 0 \end{bmatrix}, \quad (4.120)$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ -\bar{D}_{21} \\ \bar{D}_{11} \end{bmatrix}. \quad (4.121)$$

The approximate linear state-space model is defined as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \bar{D}_{22} \bar{m} g & 0 & 0 \\ -\bar{D}_{12} \bar{m} g & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\bar{D}_{21} \\ \bar{D}_{11} \end{bmatrix} T. \quad (4.122)$$

Control design

A global switching control strategy is adopted for the controller design. The two operating regions defined by this strategy are illustrated in Figure 4.54. In *Zone 1*, the control input corresponds to a swing-up controller that drives the system toward a neighborhood of the unstable equilibrium. In the *intermediate region*, no control action is applied. In *zone 2*, a delay-based stabilizing controller is activated to maintain the system around the unstable equilibrium point.

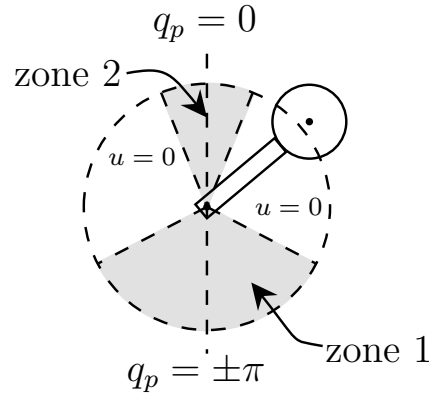


Figure 4.54 / Operating zones of the switching controller.

In the remainder of this section, we describe the design of the two controllers operating in zone 1 (*Swing-Up Control*) and zone 2 (*Stabilizing Control*).

Swing-Up control

The first objective is to drive the pendulum from its stable downward equilibrium to a neighborhood of the upright (unstable) equilibrium. Since the system is conservative, the swing-up controller is designed based on energy considerations rather than the full dynamic model. This approach regulates the system energy using a passivity-based framework combined with saturation functions.

By expressing $\ddot{q}_d$ from the model (4.97), the total energy of the system can be written as

$$E = \frac{1}{2} (m_p \ell_{cp}^2 + m_d \ell^2 + J_p) \dot{q}_p^2 + (m_p \ell_{cp} + m_d \ell) g (1 + \cos(q_p)). \quad (4.123)$$

The passivity properties of the system lead to the following expression for the time derivative of the total energy:

$$\begin{aligned} \frac{dE}{dt} &= (m_p \ell_{cp}^2 + m_d \ell^2 + J_p) \dot{q}_p \ddot{q}_p - (m_p \ell_{cp} + m_d \ell) g \dot{q}_p \sin(q_p), \\ &= -\dot{q}_p T(t). \end{aligned}$$

Assuming the pendulum undergoes continuous rotation, the unstable equilibrium occurs at $q_{p0} = 2k\pi$, where $k = 0, 1, 2, \dots, n$. The energy evaluated at this point provides the reference value for the control strategy:

$$E_0 = 2(m_p \ell_{cp} + m_d \ell) g. \quad (4.124)$$

The control input is then defined as

$$-T = -k_d (E - E_0) \text{sat}(\dot{q}_p), \quad (4.125)$$

$$\text{sat}(\dot{q}_p) = \begin{cases} d & \text{if } \dot{q}_p > \delta, \\ d & \text{if } \dot{q}_p < -\delta, \\ \dot{q}_p & \text{if } |\dot{q}_p| \leq \delta, \end{cases} \quad (4.126)$$

$$\delta = \frac{V_{\max} k_{mv}}{k_d R E_0} = \frac{k_{mi} i_{\max}}{k_d E_0}, \quad (4.127)$$

where $V_{\max}$ and $i_{\max}$ are the motor's maximum voltage and current, respectively; k_{mv} and k_{mi} are the torque-voltage and torque-current constants; R is the motor resistance; and k_d is a design constant used to tune the control response.

Delayed state feedback stabilizing control

By linearizing the nonlinear model (4.97), we derive the standard state-space representation of the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (4.128)$$

where $\mathbf{x}(t) := [q_p \dot{q}_p \dot{q}_d]^\top$ is the state vector, $\mathbf{u}(t)$ is the control input, and the matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} 0 \\ c \\ d \end{bmatrix},$$

The scalar coefficients a , b , c , and d are given by the following physical parameters of the system:

$$a := \frac{(m_p \ell_{cp} + m_d \ell)g}{m_p \ell_{cp}^2 + m_d \ell^2 + J_p}, \quad b := -a, \\ c := \frac{-1}{m_p \ell_{cp}^2 + m_d \ell^2 + J_p}, \quad d := \frac{m_p \ell_{cp}^2 + m_d \ell^2 + J_p + J_d}{(m_p \ell_{cp}^2 + m_d \ell^2 + J_p)J_d}.$$

Remark 4.14. An important practical consideration is that, in real-world applications, the full system state $\mathbf{x}(t)$ is often not directly available in real time. Instead, it must be reconstructed or estimated from delayed measurements, as discussed in [84].

In this context, we propose modifying the control law as $\mathbf{u}(t) = -\mathbf{K} \mathbf{x}(t - h)$, where $h \in \mathbb{R}_+$ represents a time delay, and $\mathbf{K} = [k_1 \ k_2 \ k_3]$ is the control gain vector.

Consequently, the stability of the closed-loop system depends on the location of the roots of the associated characteristic function $\Delta : \mathbb{C} \rightarrow \mathbb{C}$, given by

$$\Delta(s) = \det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}e^{-hs}). \quad (4.129)$$

The presence of the exponential term e^{-hs} makes Δ a retarded-type quasipolynomial, which inherently possesses an infinite number of roots. Nonetheless, its stability is determined by the location of its *spectral abscissa*.

Let us write $\Delta(s)$ in the following way

$$\Delta(s) = s^3 - as + e^{-hs}(\alpha_1 s^2 + \alpha_2 s + \alpha_3),$$

where $\alpha_1 = (dk_3 + ck_2)$, $\alpha_2 = ck_1$ and $\alpha_3 = k_3(bc - ad)$.

Similarly to delay-free approaches, specific values can be assigned to a subset of the characteristic roots. In particular, up to three roots can be placed, corresponding to the three independent control gains.

A well-known method in the literature, referred to as Multiplicity-Induced Dominancy (MID) [85], enforces high root multiplicity to ensure dominance. However, a major drawback of this approach is that highly multiple roots are extremely sensitive to small perturbations.

To overcome this limitation, more recent techniques propose the placement of multiple distinct real roots along the real axis [86].

With the above discussion in mind, it is worth noting that solving the following system of linear equations is equivalent to assigning three distinct real roots (s_1, s_2, s_3) .

$$\mathbf{M}\boldsymbol{\alpha} = \mathbf{S}, \quad (4.130)$$

where

$$\mathbf{M} := \begin{bmatrix} s_1^2 e^{-hs_1} & s_1 e^{-hs_1} & e^{-hs_1} \\ s_2^2 e^{-hs_2} & s_2 e^{-hs_2} & e^{-hs_2} \\ s_3^2 e^{-hs_3} & s_3 e^{-hs_3} & e^{-hs_3} \end{bmatrix}, \quad \boldsymbol{\alpha} := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, \quad \mathbf{S} := \begin{bmatrix} s_1^3 - as_1 \\ s_2^3 - as_2 \\ s_3^3 - as_3 \end{bmatrix}.$$

In this case, to simplify the analysis, we propose $s_2 = 2s_1$, and $s_3 = 3s_1$. That is, the roots are equally spaced from one another.

PROPOSITION
4.30

The closed-loop system (4.128) with the delayed feedback control law $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t-h)$, admits the placement of three equidistant characteristic roots at $(s_1, 2s_1, 3s_1)$ for any $s_1 \neq 0$. Moreover, the corresponding control gains are given by

$$k_1^* = \frac{\alpha_2^*}{c}, \quad (4.131)$$

$$k_2^* = \frac{\alpha_1^* - ck_3^*}{d}, \quad (4.132)$$

$$k_3^* = \frac{\alpha_3^*}{bc - ad}, \quad (4.133)$$

where

$$\alpha_1^* = \frac{e^{hs_1} \left(-4(a - 4s_1^2)e^{hs_1} + 3(a - 9s_1^2)e^{2hs_1} + a - s_1^2 \right)}{2s_1}, \quad (4.134)$$

$$\alpha_2^* = \frac{5}{2} (s_1^2 - a)e^{hs_1} + 8(a - 4s_1^2)e^{2hs_1} - \frac{9}{2} (a - 9s_1^2)e^{3hs_1}, \quad (4.135)$$

$$\alpha_3^* = 3s_1 e^{hs_1} \left(a(e^{hs_1} - 1)^2 - s_1^2 (-8e^{hs_1} + 9e^{2hs_1} + 1) \right). \quad (4.136)$$

Proof. The proof follows directly from the fact that for any $s_1 \neq 0$, choosing $s_2 = 2s_1$ and $s_3 = 3s_1$, ensures that the determinant of the matrix $\det(\mathbf{M}) = -2s_1^3 e^{-6hs_1} \neq 0$. Since $\mathbf{M}$ is invertible, the system of equations for the control gains has a unique solution, confirming the desired pole placement. $\square$

Proposition 4.30 ensures that, by properly selecting the control gains $\mathbf{K}$, it is possible to enforce $\Delta(s_1) = \Delta(2s_1) = \Delta(3s_1) = 0$. However, as previously mentioned, Δ has infinitely many characteristic roots. Therefore, it is also necessary to verify that, for $s_1 < 0$, the root s_1 corresponds to the spectral abscissa, that is, it is the dominant root of the system.

Switching controller

The switching logic between the two designed controllers is given by the following law:

$$T = \begin{cases} k_d(E - E_o) \text{sat}(\dot{q}_1) & \text{if } (q_p - \hat{x}_1)^2 + \dot{q}_p^2 \geq \delta, \\ -\mathbf{K}\mathbf{x}(t-h) & \text{if } (q_p - \hat{x}_1)^2 + \dot{q}_p^2 < \delta, \end{cases} \quad (4.137)$$

where $\hat{x}_1 = \pm 2k\pi$ for $k = 0, 1, 2, \dots$. This switching criterion is designed to activate the appropriate controller based on the system's proximity to an unstable equilibrium point, located at $(q_p, \dot{q}_p) = (2k\pi, 0)$ (see Figure 4.54), such as $(0, 0)$ or $(2\pi, 0)$.

A key feature of this strategy is that the parameter δ directly influences the pendulum's angular velocity during the switching phase. By appropriately tuning δ , this velocity can be significantly reduced, thereby decreasing the peak torque required to capture the pendulum.

Numerical results

The closed-loop system was numerically simulated in MATLAB R2023a using the `dde23` solver. The simulation source code is available in the following Zenodo repository: <https://doi.org/10.5281/zenodo.15829866>.

For the numerical results, the masses and inertias of both the pendulum and the disk were set as follows:

Inertia	Mass	Length
J_p	m_p	ℓ_{cp}
$357.56 \cdot 10^{-6} \text{ kg m}^2$	$143.8 \cdot 10^{-3} \text{ kg}$	$57.15 \cdot 10^{-3} \text{ m}$
J_d	m_d	ℓ
$358.71 \cdot 10^{-6} \text{ kg m}^2$	$278 \cdot 10^{-3} \text{ kg}$	$114.3 \cdot 10^{-3} \text{ m}$

Table 4.7 / Systems parameters for Inertia Wheel Pendulum.

By setting $s_1 = -1$ and a delay of $h = 10 \text{ ms}$, and using the pole assignment formulas (4.131)–(4.133), the resulting control gains are given by

$$\mathbf{K} = [-4.38444 \times 10^{-1}, -2.97807 \times 10^{-2}, -2.28586 \times 10^{-5}].$$

With the linear controller tuned, a velocity analysis of the pendulum is then performed to evaluate the speed at which it transitions from the stable equilibrium to the unstable one. This analysis is carried out by adjusting the gain k_d .

Example 4.12

The closed-loop system response is shown in Figure 4.55 for $k_d = 1$.

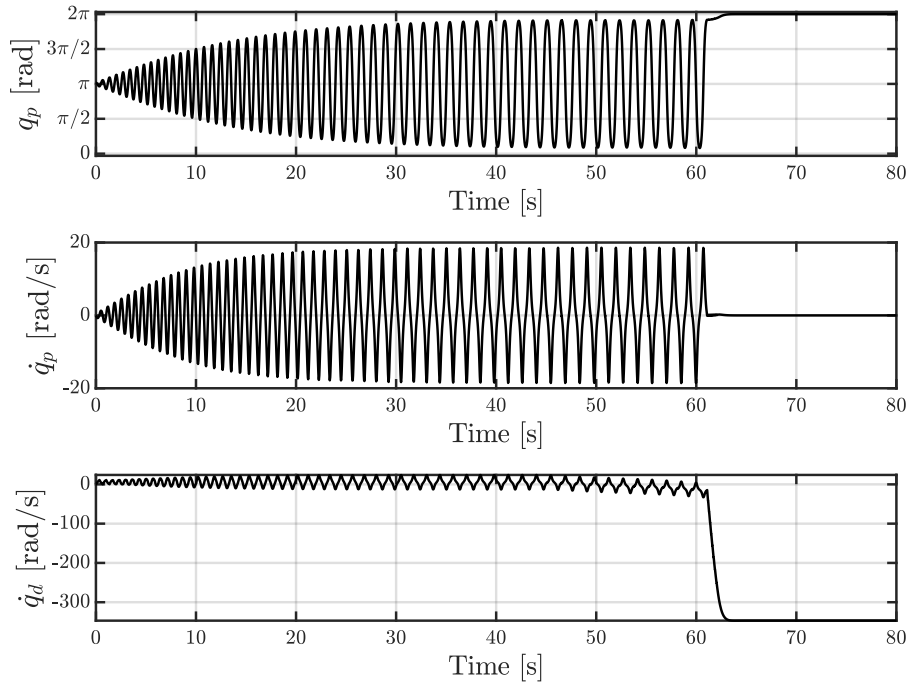


Figure 4.55 / Closed-loop system's response with $k_d = 1$.

To further illustrate the system behavior, Figure 4.56 presents the phase portrait of the system states. This diagram provides insight into the trajectories followed by the system in the state space, highlighting the transition from the stable equilibrium to the unstable one.

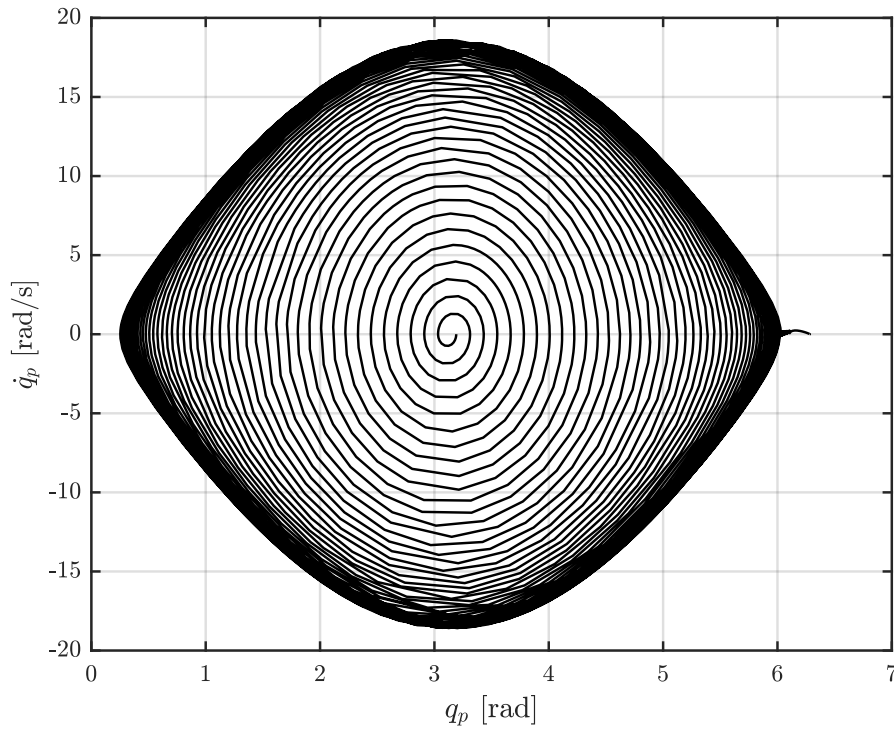


Figure 4.56 / Closed-loop system's phase diagram with $k_d = 1$.

Example 4.13

On the other hand, by increasing the value of the gain $k_d = 5$, it is observed how the system passes faster from the roll control to the stabilization control, as shown in Fig 4.57.

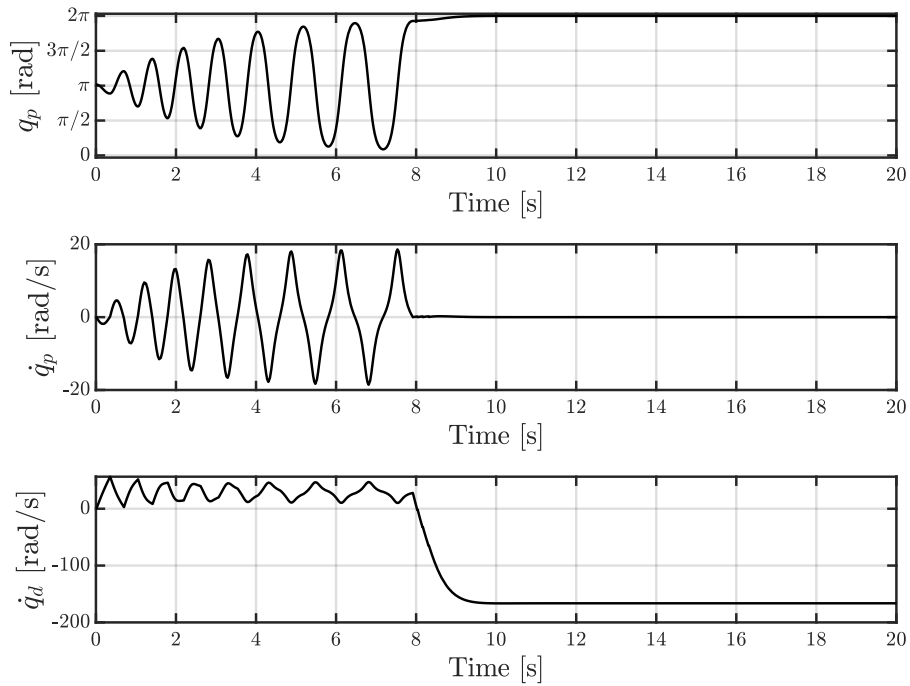


Figure 4.57 / Closed-loop response obtained with $k_d = 5$.

Additionally, Figure 4.58 shows the phase portrait of the system states for $k_d = 5$. Compared to the previous case, the trajectories indicate a faster transition from the roll control phase to the stabilization phase.

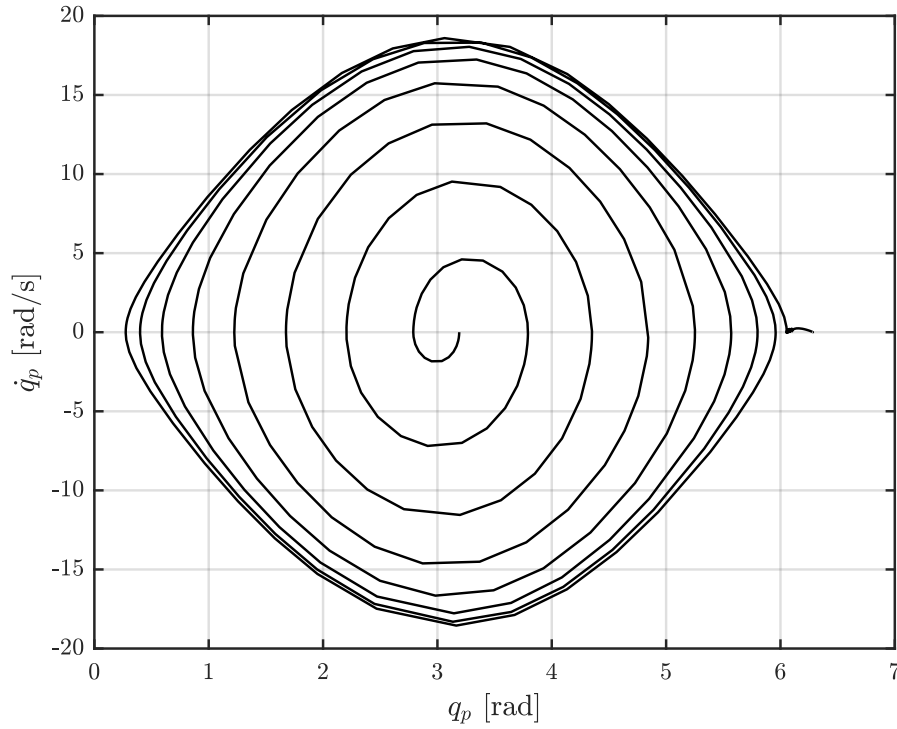


Figure 4.58 / Closed-loop system's phase diagram with $k_d = 5$.

Experimental results

The mechanical structure used in this work (see Figure 4.59) consists of a fixed base connected to two parallel lateral supports by means of removable fasteners (screws), arranged perpendicularly. An axis passes through the side supports, allowing the pendulum to swing back and forth, defining the degree of freedom q_p , and is connected to the pendulum via a coupling.

At the opposite end of the pendulum, a DC motor is mounted, and an inertia disk is attached to its shaft, thereby forming the actuated degree of freedom q_d .

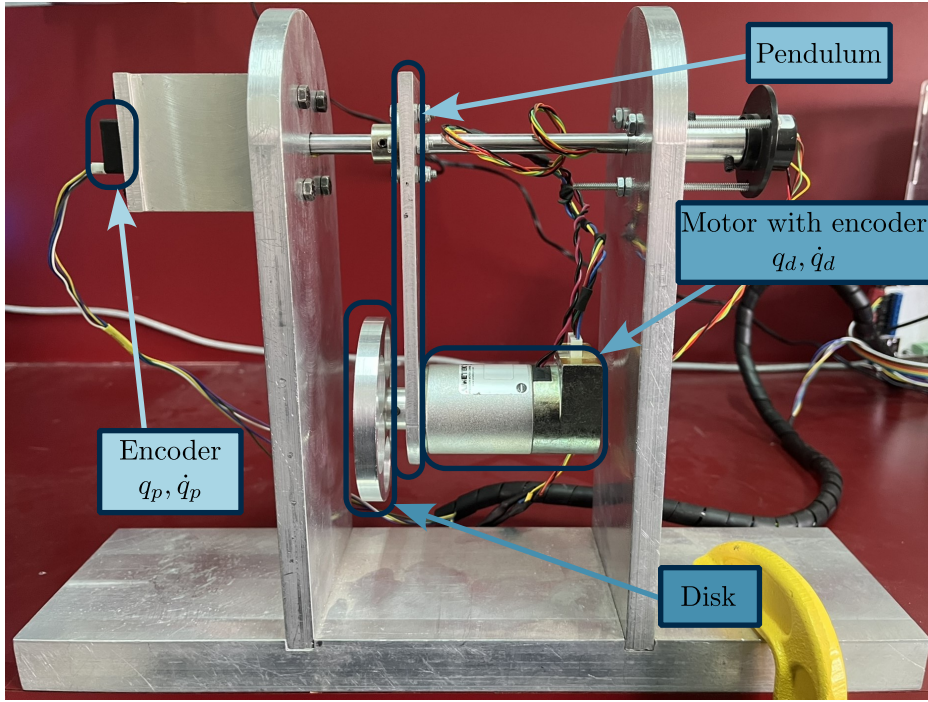


Figure 4.59 / Experimental setup for IWP.

For the implementation of reliable communication between the computer and the mechanical system, a NI myRIO-1900 was used. The control algorithm was programmed in LabVIEW, with a constant sampling time of $t_s = 10$ ms.

All experiments were conducted under the same initial conditions $[q_p \ \dot{q}_p \ \dot{q}_d]^\top = [\pi \ 0 \ 0]^\top$, which correspond to the pendulum starting from its stable downward equilibrium (rest position). As discussed in the *Swing-up control* section, since the controller is energy-based, a 250 ms excitation routine is implemented to overcome the system's initial inertia and initiate the pendulum motion.

To determine an appropriate value of k_d , several experimental trials were performed. The selection criterion was to choose k_d sufficiently small to ensure that the pendulum reaches the upright position with low angular velocity, yet large enough to guarantee that the transition is successfully completed. If the disk reaches excessive speeds, it becomes unable to capture the pendulum, a situation that may appear in simulations due to the absence of hardware constraints.

Based on these considerations, the experimental tests were conducted with $k_d = 0.45$.

Experiment 4.2

The first experiment evaluated only the linear controller. In this case, the experiment was initiated with the pendulum manually driven to its unstable equilibrium position, reaching this configuration at approximately $t = 4$ s, where the delayed control law was activated. The results of this experiment

are presented in Figure 4.60.

Additionally, a video of the experimental implementation is available at <https://youtube.com/shorts/aWbqrR3LeTM?feature=share>.

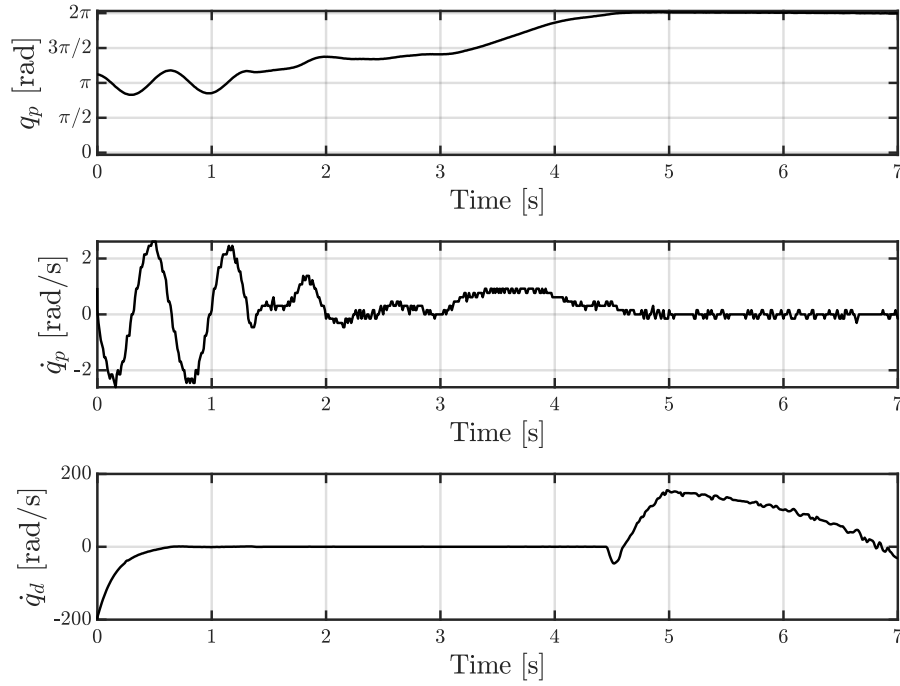


Figure 4.60 / Experiment 1.

Figure 4.61 presents the phase portrait of the system states obtained from the experimental data. This representation allows a direct comparison between the experimental behavior and the numerical simulations, providing insight into the practical implementation and performance of the proposed control strategy.

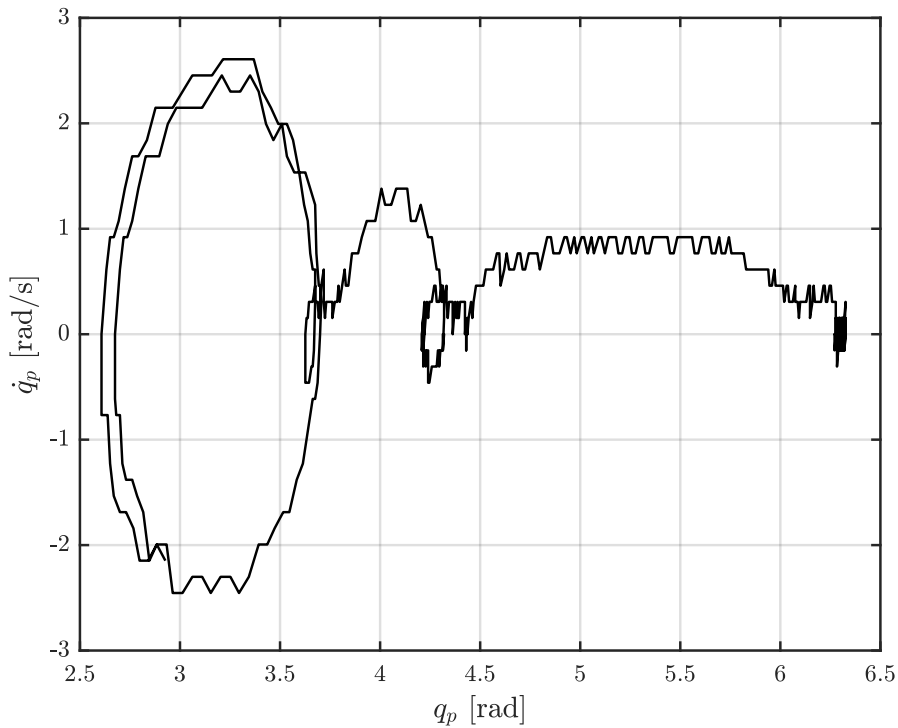


Figure 4.61 / Closed-loop phase portrait corresponding to Experiment 1.

Experiment 4.3

As a second test, the controller was activated while the system started from its stable downward equilibrium position. This experiment was conducted to evaluate both the swing-up and stabilization controllers under realistic operating conditions.

Figure 4.62 illustrates the action of both control stages. In particular, the swing-up controller requires approximately 30 s to drive the pendulum toward the upright position. Once the pendulum reaches a neighborhood of the unstable equilibrium, the stabilization controller is activated, maintaining the pendulum around the vertical position.

A video of this experiment is available at

<https://youtube.com/shorts/16Woizy9Hss?feature=share>.

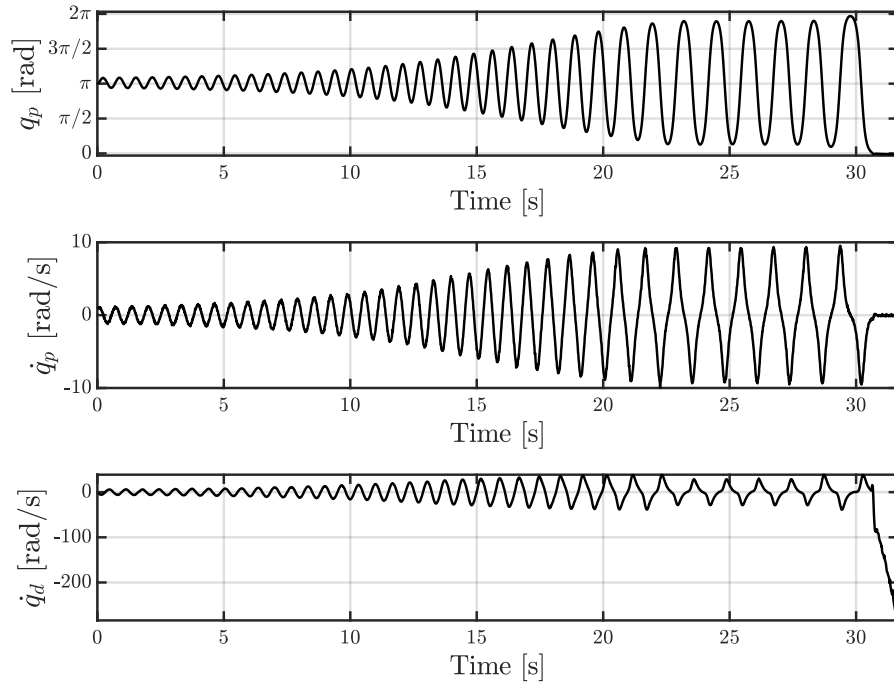
**Figure 4.62** / Experiment 2.

Figure 4.63 illustrates the phase portrait of the system states corresponding to the second experiment. This visualization provides additional validation of the controller's effectiveness under real-world operating conditions.

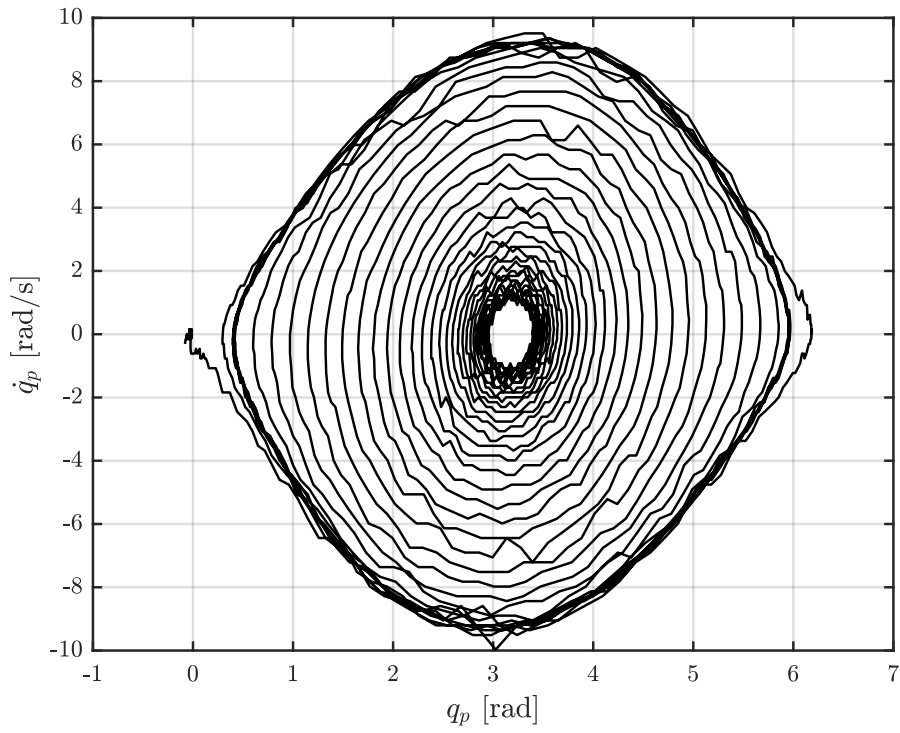


Figure 4.63 / Closed-loop system's phase diagram for experiment 2.

5 Delay-Based Controllers for Time-Delay Systems

The scientific man does not aim at an immediate result.

Nikola Tesla

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This chapter studies the stabilization and performance analysis of time-delay systems using delay-based control structures. First, the Integral Retarded (IR) controller is analyzed through the characterization of stability crossing boundaries and admissible controller regions. Then, an analytical–metaheuristic framework is introduced to optimize controller parameters according to performance criteria such as maximum exponential decay and control effort. The chapter also introduces the Integral–Proportional Delayed (I-P δ) controller, deriving explicit stability conditions and σ -stability regions for general delayed plants. Several numerical and experimental examples demonstrate the effectiveness of the proposed methodologies.

5.1 Integral Retarded Controller (IR)

The control and analysis of DC motors are common tasks in various engineering and automation applications. Managing the velocity of a DC motor based on voltage input is a classic control problem addressed using the type-0 transfer function.

In this section, we explore the first-order transfer function, which is the simplest in the control area.

$$G_p(s) := \frac{Y(s)}{V(s)} = \frac{K}{s + a}, \quad K, a \in \mathbb{R}. \quad (5.1)$$

In this equation, K represents gain, and a represents the natural resistance to changes in speed.

However, this study study focuses on a system characterized by an inherent time-delay τ . This delay is constant and integrated into the system’s dynamic model.

Considering the known delay factor, the system can be described as transitioning to a state that includes the effects of this latency. The influence of the delay on the system's behavior can be explained through the modified transfer function

$$G(s) := G_p(s)e^{-\tau s} = \frac{K}{s+a} e^{-\tau s}, \quad \tau > 0. \quad (5.2)$$

Therefore, designing a controller is crucial in accounting for and counteracting the time-delayed effects within the system. This aspect of control design poses significant challenges as it entails striking a delicate balance between compensating for the delay while maintaining overall system performance.

The design of the IR controller, originally posited by [36], omits unintentional delays within the control loop, an assumption permissible when such delays are negligible. Considering that the controller and the plant are well-known in the literature, this section and the rest of the paper employ a fixed unintentional measurement delay, $\tau > 0$, in the IR to manage the system $G(s)$. The IR controller is defined by

$$C(s) := \frac{k_i - k_r e^{-hs}}{s}, \quad k_i, k_r \in \mathbb{R}, h > 0, \quad (5.3)$$

where k_i represents the integral action, k_r the delayed action and h the delay of the controller. The controller system, as depicted in Fig. 5.1, illustrates the IR controller's structure. It shows the standard integral component, which assures error correction over time, and the retarded component, which introduces a calculated delay, counterbalancing the delay inherent in the system's measurements.

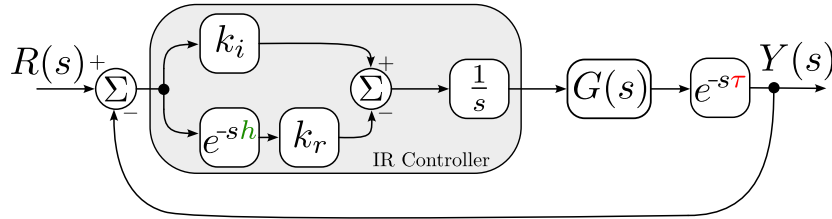


Figure 5.1 / Single-loop control scheme using the IR controller and a delay measurement at the output of the process.

An advantage of employing this controller is its ability to ensure a consistent and steady error even when subjected to step-type inputs. Conversely, the integrator that influences both control actions (k_i, k_r) functions as a filter. As a result, this term filters out external disturbances or high-frequency noise in the data.

Analytical approach

The closed-loop system depicted in Fig. 5.1 is described by the transfer function

$$H_{yr} := \frac{Y(s)}{R(s)} = \frac{Kk_i - Kk_r e^{-hs}}{s^2 + as + Kk_i e^{-\tau s} - Kk_r e^{-(\tau+h)s}}. \quad (5.4)$$

To analyze the system's stability, one must examine the characteristic quasi-polynomial $\Delta_{yr}(s) : \mathbb{C} \times \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{C}$

$$\Delta_{yr}(s; k_i, k_r, h) := s^2 + as + Kk_i e^{-\tau s} - Kk_r e^{-(\tau+h)s}. \quad (5.5)$$

Identifying the system's stability zones is the initial step in controller tuning. This step is vital to apply a meta-heuristic approach that efficiently searches within the range of stabilizing controller parameters.

Remark 5.1. In the absence of delay ($\tau = 0$), the tuning problem adheres to the solutions presented in [36], offering analytical expressions to define the parameters k_i , k_r , and h . This tuning-rule obtains a triple real root in $-\sigma$, resulting in the maximum exponential decay (MED). Consequently, any unintentional delay within the feedback loop presents a persistent challenge for these systems. ┘

Proceeding with the assumption that $\tau > 0$. The system described in (5.4) is categorized as a retarded-type due to the higher-order variable coefficient not being multiplied by exponentials, i.e., the absence of s^2 in the exponential terms. This classification is significant, as retarded-type systems are known to demonstrate continuous root displacement in response to parameter variations [87]. As such, the roots cannot cross from the LHP to the RHP without first encountering stability boundaries. Several foundational definitions are provided to address the stability of such systems.

DEFINITION

5.1

The *frequency crossing set* $\Omega \subset \mathbb{R}$ is the set of all ω for which there exists at least one triplet (k_i, k_r, h) in $\mathbb{R}^2 \times \mathbb{R}_+$ such that

$$\Delta_{yr}(i\omega; k_i, k_r, h) = 0. \quad (5.6)$$

Remark 5.2. The numerator and denominator of (5.4) as polynomials with real coefficients satisfy the relation

$$\Delta_{yr}(-i\omega; k_i, k_r, h) = \overline{\Delta_{yr}(i\omega; k_i, k_r, h)}. \quad (5.7)$$

In other words, the above equality states that $\Omega \subset \mathbb{R}_+ \cup \{0\}$. ┘

DEFINITION

5.2

For a fixed $h > 0$, the *stability crossing curves (stability crossing boundaries)* will be denoted by $\mathcal{K}$, and is defined as the set of all parameters $(k_i, k_r) \in \mathbb{R}^2$ for which there exists at least one $\omega \in \Omega$ satisfying $\Delta_{yr}(i\omega; k_i, k_r, h) = 0$. Moreover, any point $\mathbf{k} \in \mathcal{K}$ is known as a *crossing point*.

Characterizing the stability crossing curves is essential for control design. Stability may be compromised or enhanced in two scenarios: a root crossing at the CRB or the RRB. In this approach, we maintain a fixed value of the intentional delay $h > 0$ to find the stability boundaries in the parameter plane (k_i, k_r) .

PROPOSITION

5.1

Let $h > 0$ be a constant value, and define $\Omega := \bigcup_{\ell} \Omega_{\ell}$ for $\ell \in \mathbb{N} \cup \{0\}$ where

$$\Omega_{\ell} := \left\{ \omega \in \mathbb{R}_+ \mid \omega \in \left(\frac{\pi}{h} \ell, \frac{\pi}{h} (\ell + 1) \right) \right\}.$$

Then, $\omega \in \Omega$ is a crossing frequency iff $\mathbf{k}(\omega) := [k_i(\omega) \ k_r(\omega)]^{\top}$ and $\mathbf{k}(\omega) \in \mathcal{K}_{\omega}$, where

$$k_i(\omega) := \frac{\omega}{K \sin(h\omega)} (-a \cos(\delta\omega) + \omega \sin(\delta\omega)), \quad (5.8a)$$

$$k_r(\omega) := \frac{\omega}{K \sin(h\omega)} (-a \cos(\tau\omega) + \omega \sin(\tau\omega)), \quad (5.8b)$$

where $\delta = \tau + h$.

Proof. The proof follows straightforwardly by noting that if $\Delta_{yr}(\mathbf{i}\omega; k_i, k_r, h) = 0$ then

$$\Re\{\Delta_{yr}(\mathbf{i}\omega; k_i, k_r, h)\} = 0,$$

$$\Im\{\Delta_{yr}(\mathbf{i}\omega; k_i, k_r, h)\} = 0.$$

□

PROPOSITION

5.2

For a fixed delay $h > 0$, the line $\mathbf{k}_0 \in \mathcal{K}_0$ is described by

$$\mathbf{k}_0 = \left\{ [k_{r0} \ k_{r0}]^{\top} \mid k_{r0} \in \mathbb{R} \right\}, \quad (5.9)$$

which represents a root crossing at the origin of the complex plane.

Proof. This proposition is supported analogously to the CRB, focusing on the condition where $\Delta_{yr}(0; k_i, k_r, h) = 0$. □

Remark 5.3. The set $\mathcal{K}$ consists of locations that correspond to a complex crossing ($\mathcal{K}_{\omega}$) or a crossing by the origin ($\mathcal{K}_0$), as shown by the definitions above. Therefore, it can be concisely stated that the limit of these crossings is $\mathcal{K} = \mathcal{K}_{\omega} \cup \mathcal{K}_0$. ┘

Example 5.1

Let the system be defined by the following transfer function

$$G(s) = \frac{1}{s+1} e^{-\frac{3}{4}s} \quad (5.10)$$

then the regions of stability are obtained using the Propositions 5.1-5.2

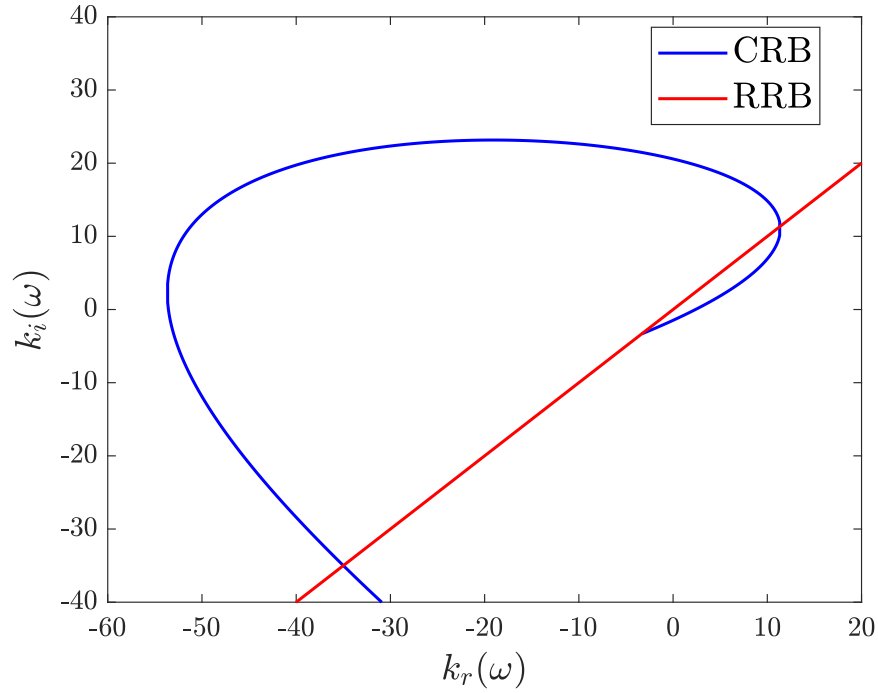


Figure 5.2 / Stability crossing curves for system (5.10).

Remark 5.4. For σ -stability analysis, i.e. when $s = -\sigma + i\omega$, $\sigma \in \mathbb{R}_+$ it is recommended to study of [88].

Metaheuristic-Analytical approach

As discussed in [88], the delay is considered as a fixed parameter, while the remaining gains of the IR controller are adjusted. The design of controllers based on performance criteria such as maximum exponential decay (MED), control effort, or settling time is not straightforward from an analytical point of view.

One practical alternative is to define multi-objective cost functions that combine different performance measures into a single design criterion. However, when meta-heuristic optimization methods are applied, an important difficulty arises in defining an appropriate search space for the controller gains. This selection usually depends on the designer's experience and, if not properly chosen, may lead to non-optimal solutions.

To address this issue, a combined strategy is proposed. First, an analytical approach is used to restrict the search space in order to guarantee stability. Then, a meta-heuristic method is applied within this region to find the controller gains that optimize the selected cost function. In this way, the optimization process is performed over a stable set of solutions, improving both reliability and computational efficiency.

The optimization problem can be stated as follows:

$$\begin{aligned} & \text{minimize} && J \\ & \text{subject to} && \boldsymbol{\kappa} \in \mathcal{S}, \end{aligned} \quad (5.11)$$

where $\boldsymbol{\kappa} := [k_p^* \ k_i^*]^\top$ with $h > 0$, or $\boldsymbol{\kappa} := [k_p^* \ k_i^* \ h^*]^\top$ depending on the case. The set $\mathcal{S}$ represents the admissible region of controller gains. Figure 5.3 shows a geometric interpretation of this optimization problem in the gain space. The shaded region, denoted by $\mathcal{S}$, corresponds to the set of gains that ensure system stability. Inside this region, the point associated with $\min J$ indicates the optimal combination of gains, where the cost function reaches its minimum value under the imposed constraints. This representation helps to understand how the search for optimal gains is carried out within the stable region. The shaded region represents

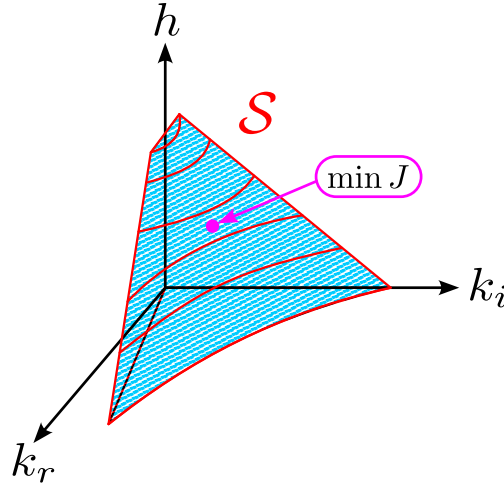


Figure 5.3 / Geometric Visualization of Gain Space Optimization.

the permissible gain space $\mathcal{S}$, with the highlighted point indicating the optimal solution where the cost function J is minimized.

Two cost functions are considered in this work. The first one, J_1 , is defined to maximize the MED by finding suitable values of (k_i, k_r, h)

$$J_1 = -\sigma, \quad \sigma \in \mathbb{R}_+. \quad (5.12)$$

The objective is to move the dominant poles (spectral abscissa) of the system further to the left in a local region, which increases the decay rate and improves the speed response.

The second cost function, J_2 , is designed to achieve a compromise between fast response and low control effort over the time interval from the initial condition ($t_0 = 0$) to the settling time t_s

$$J_2 = t_s(\alpha) + \int_0^{t_s} (\Psi u(\tau)^2 + \Upsilon \dot{u}(\tau)^2) d\tau, \quad \alpha, \Psi, \Upsilon \in \mathbb{R}_+. \quad (5.13)$$

This function aims to obtain a controller that provides a fast response while keeping the control energy within reasonable limits.

To solve the optimization problem, Algorithm 2 is proposed. This procedure combines the stability boundaries, the definition of the cost functions, and the use of a main optimization technique, such as a genetic algorithm, to search for the optimal solution.

Algorithm 2 Synergetic optimization procedure for controller gains selection.

Require: $\mathcal{K}_\omega, \mathcal{K}_0, \text{opt}, \Psi, \Upsilon$;

Ensure: Solution sol;

1: **Define:**

$$\mathcal{K}_\omega \leftarrow [k_i(\omega), k_r(\omega)]^T;$$

$$\mathcal{K}_0 \leftarrow [k_{r0}, k_{r0}]^T, \quad k_{r0} \in \mathbb{R};$$

2: Compute $\mathcal{K} \leftarrow \text{D_METHOD}(\mathcal{K}_\omega, \mathcal{K}_0)$;

$$\mathcal{K} \leftarrow \mathcal{K}_\omega \cup \mathcal{K}_0;$$

3: Define cost function J ;

4: **if** opt == 1 **then**

5: $J \leftarrow -\sigma$;

6: **else if** opt == 2 **then**

$$J \leftarrow t_s(\alpha) + \int_0^{t_s} (\Psi u(\tau)^2 + \Upsilon \dot{u}(\tau)^2) d\tau;$$

7: **end if**

8: **Main procedure:**

9: Define frequency grid $\omega = 0^+ : \omega_s : \omega_f$;

10: Compute $\mathcal{K} \leftarrow \text{D_METHOD}(\omega)$;

11: $[k_i^l, k_r^l]^T \leftarrow \min_\omega(\mathcal{K})$;

12: $[k_i^u, k_r^u]^T \leftarrow \max_\omega(\mathcal{K})$;

13: Compute:

$$\text{sol} \leftarrow \text{GA}(\kappa, J, 2, [], [], [], [k_i^l, k_r^l], [k_i^u, k_r^u]);$$

14: **return** sol;

Remark 5.5. In this work, a genetic algorithm is used as the main optimization tool. However, other methods such as particle swarm optimization, least squares, or gradient-based techniques can also be applied to solve similar problems. ┘

Remark 5.6. Since the feasible domain is compact and the cost function is continuous with respect to its parameters (and convex), the existence of a global minimum is guaranteed. This follows from the Bolzano–Weierstrass Theorem (see [66] for further details). ┘

Using the stability conditions for the IR controller shown in the Propositions 5.1–5.2 in closed-loop with the system (3.35), the stability region shown in Figure 5.4 is constructed.

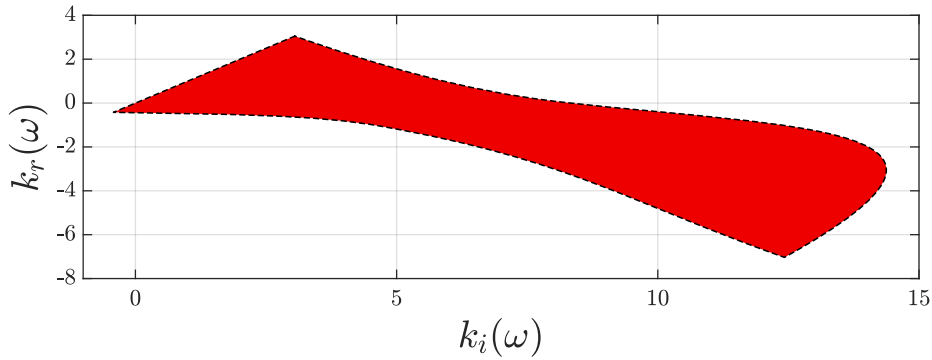


Figure 5.4 / Numerical stability chart boundary with $h = 0.1$ s.

The illustrated region denotes the stability boundaries determined by the derived stability conditions.

The genetic algorithm is then applied within this bounded and well-defined region, limiting the search space for the controller gains. In this way, all selected solutions satisfy the stability requirements of the system.

Experimental results

The benchmark tests were carried out using a square wave reference signal with an amplitude of 15 rad/s and a period of 5 seconds. This setup provides a consistent basis for comparing different controller configurations.

Remark 5.7. When $h = 0$, the proposed IR controller behaves similarly to a PI controller, although no zero is introduced at $-\frac{k_i}{k_p}$. For the case study considered here, a detailed solution using the synergetic approach can be found in [14].

Experiment 5.1

The aim of the first experiment is to improve the closed-loop performance by maximizing σ , or equivalently minimizing $-\sigma$, in order to move the dominant poles as far as possible to the left in the complex plane. This increases the exponential decay rate of the system. The optimization process was carried out by adjusting the IR controller parameters (k_i, k_r, h) . The resulting optimal values are $k_i = 2.3394$, $k_r = 1.4666$, and $h = 0.0466$, which place the dominant poles at $s_1 = -22.7660$ and $s_{2,3} = -22.7616 \pm 4.3096i$. This result confirms that a maximum value of $\sigma = 22.7616$ is achieved, with a delay value greater than the minimum sampling time of the system.

As a comparison, a classical tuning procedure for a Proportional-Integral (PI) controller is also considered. In this case, the controller gains are selected to maximize the exponential decay rate, resulting in $k_p = 0.4931$ and $k_i = 18.5254$, with a dominant pole located at $s = -118.634$.

Figure 5.5 presents the experimental results^a obtained with both controllers. The results show that Algorithm 1 effectively reduces oscillations in a servo problem. In contrast, the PI controller exhibits a more sensitive behavior, even when it is tuned to maximize the exponential decay. The settling times within the specified tolerance are approximately $t_s \approx 0.4$ s for the PI controller and $t_s \approx 1.2$ s for the IR controller. Although the IR controller presents some damped oscillations during regulation, the response remains stable. Additionally, the control signal shows voltage peaks when changes in the reference signal occur.

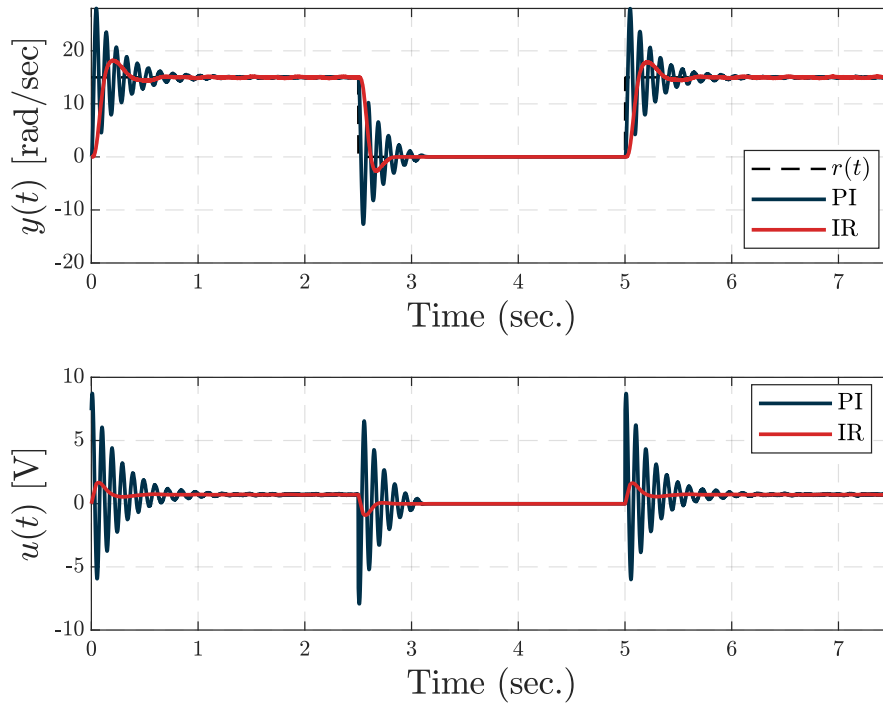


Figure 5.5 / System Response for Cost Function J_1 . The temporal response of the system with the controller optimized for J_1 , demonstrating the achieved settling time and control signal characteristics.

^aFor a detailed demonstration of the experiments, see <https://youtu.be/p6nmAHRkXs0>.

Experiment 5.2

The second experiment aims to reduce the settling time within a 1% tolerance band, similar to the MED criterion, while also taking into account the control effort and the smoothness of the control signal. For this purpose, the cost function includes two weighting factors, which allow adjusting the importance of response speed, control effort, and signal smoothness during the optimization process.

Table 5.1 presents the controller parameters obtained offline before performing the experiments shown in Figure 5.6. The results indicate that increasing Ψ

reduces the amplitude of the control signal, but also slows down the convergence to the reference. On the other hand, higher values of Υ produce a smoother control signal, with similar behavior observed across the different responses.

Controller			k_i	k_r	h [s]	σ
Name	Ψ	Υ				
IR ₁	1	1	0.6532	0.3046	0.1	7.6825
IR ₂	2	1	0.6116	0.2791	0.1	8.4531
IR ₃	1	10	0.5860	0.2642	0.1	9.3234
IR ₄	10	1	0.5415	0.2481	0.1	8.7143
IR ₅	50	1	0.5243	0.3464	0.1	2.6643

Table 5.1 / Resulting parameters in various controllers for J_2 , it is worth noting that Ψ and Υ weights parameters of the cost function are different starting points in the controller algorithm.

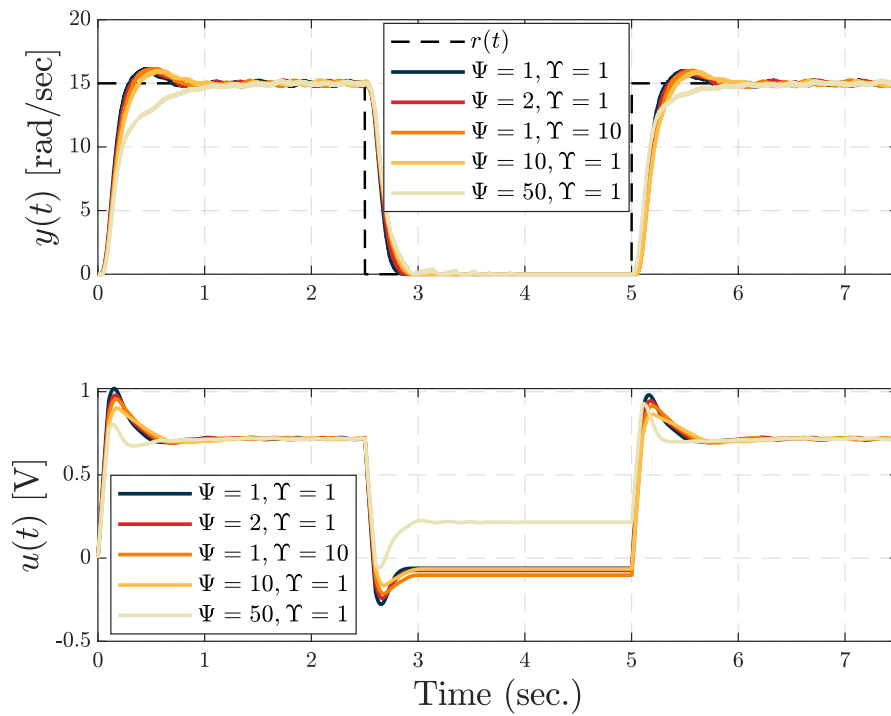


Figure 5.6 / System Response for Cost Function J_2 . The graph showcases the system's performance when optimized for the multi-objective cost function J_2 , reflecting the trade-offs between response speed and control signal smoothness.

5.2 Integral Proportional Delayed Controller (I-P δ)

Problem formulation

Consider the class of strictly proper LTI SISO plants with input–output delays, described by the transfer function

$$G(s) = \frac{P(s)}{Q(s)} e^{-\tau s}, \quad (5.14)$$

where $P(s)$ and $Q(s)$ are polynomials with real coefficients, and $\tau \geq 0$ denotes the inherent delay of the plant. To ensure the well-posedness of the stability analysis and the causality of the closed-loop system, the following regularity assumptions are introduced.

ASSUMPTIONS

5.1

The plant $G(s)$ satisfies the following structural properties:

- (i) $G(s)$ is strictly proper, i.e., $\deg(Q(s)) > \deg(P(s))$.
- (ii) $P(s)$ and $Q(s)$ are coprime polynomials.
- (iii) $|P(i\omega)| > 0$, $\forall \omega \in \mathbb{R}$.

Assumption 5.1-(i) guarantees that the resulting closed-loop system belongs to the class of retarded-type systems. This property ensures that the system possesses only finitely many characteristic roots in any right half-plane, allowing the stability analysis to be characterized by a finite set of dominant roots.

Assumption 5.1-(ii) prevents pole–zero cancellations, implying that input–output stability is equivalent to the internal stability of the corresponding state-space realization. Finally, Assumption 5.1-(iii) is introduced mainly to simplify the presentation of the subsequent results.

Within the class of linear systems, time-delay systems, i.e., systems with delays in their inputs, outputs, or internal dynamics, have received significant attention in the literature. Several authors, including [89] [90] [91], have investigated the stabilization of delayed SISO systems using PID-type controllers.

An alternative approach consists of stabilizing delay-free systems by intentionally introducing delays within the controller structure. This strategy has been explored in works such as [8] [31].

In this work, we investigate a variant of delay-based controllers in which the time delay is incorporated as an additional degree of freedom in the controller tuning process. By intentionally introducing delayed dynamics into the control law, the closed-loop response can be significantly modified, enabling the development of stability criteria based on classical linear systems analysis, particularly through the frequency-domain study of the characteristic roots of the closed-loop system.

Problem 5.1: Stabilization

The objective is to stabilize the plant (5.14) using an I-P δ controller. The proposed control structure is shown in Figure 5.7. In this scheme, the integral action is placed in the forward path to ensure zero steady-state error, while the proportional and delayed terms are introduced through the feedback path to improve the transient response. This structure provides a lead-lag effect without relying on derivative action, thus avoiding the noise amplification typically associated with classical derivative controllers.

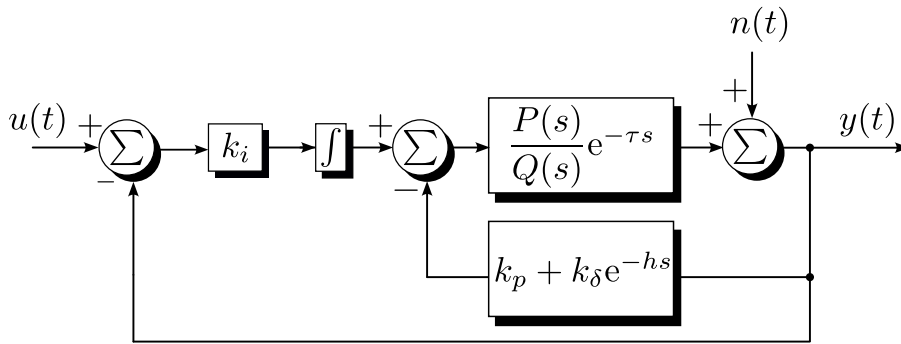


Figure 5.7 / Closed-loop control scheme with I-P δ structure.

An I-P δ controller is proposed as a modification of the classical I-PD structure introduced in [92]. The corresponding time-domain control law is given by

$$u(t) = k_i \int_0^t e(\theta) d\theta - k_p y(t) - k_\delta y(t - h), \quad (5.15)$$

where $e(t) = r(t) - y(t)$ denotes the tracking error, $\mathbf{k} = (k_p, k_i, k_\delta)$ is the vector of controller gains, and $h > 0$ is the intentionally introduced delay.

The main motivation for considering this controller structure is illustrated through the following example.

Example 5.2

Consider the third-order delayed system

$$G(s) = \frac{2s + 1}{s^3 + 3.5s^2 + 3.5s + 1} e^{-0.5s}, \quad (5.16)$$

which is open-loop stable. Two controllers were arbitrarily tuned, one with an I-PD structure and the other with an I-P δ structure, only ensuring closed-loop stability in both cases. In addition, continuous white noise was added to the measured output signal in order to emulate a noisy measurement environment. The considered noise process has zero correlation time, flat power spectral density, and infinite total energy.

The simulation results are shown in Figure 5.8.

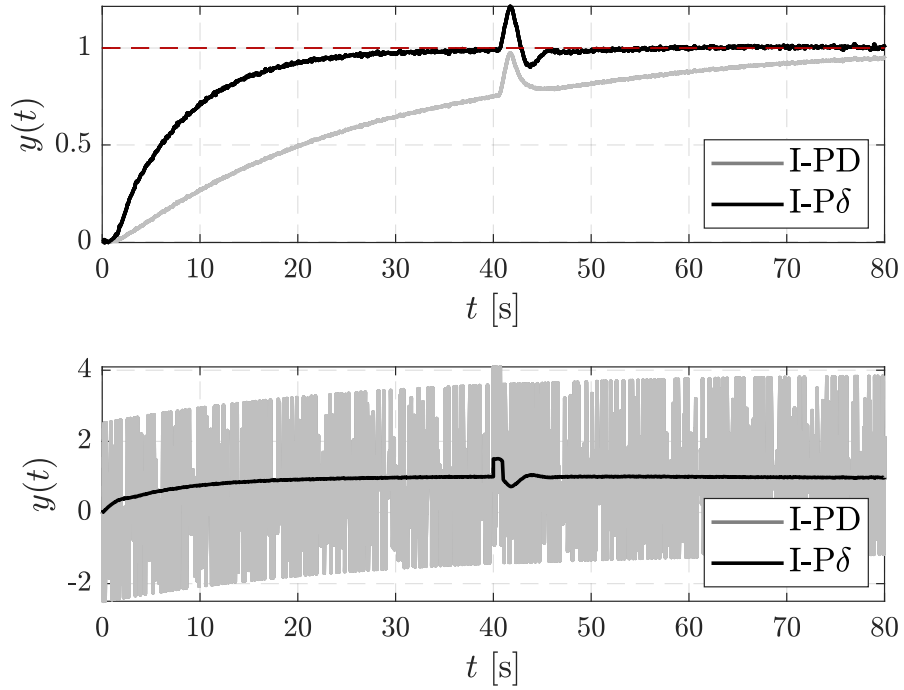


Figure 5.8 / Time responses and control efforts for the I-PD and I-P δ controllers.

Figure 5.8 presents the corresponding time responses and control efforts for both controller structures. A detailed inspection reveals that the derivative-based controller exhibits a slower response compared to the delay-based controller. Furthermore, the control effort associated with the I-PD controller is significantly amplified by the measurement noise. It is worth mentioning that the control signal was saturated at ± 2.5 , since otherwise the derivative action would generate unrealistically large control amplitudes.

In contrast, the filtering properties introduced by the delayed term can be clearly observed in the I-P δ controller. Despite the presence of measurement noise, the resulting control effort remains bounded and practically implementable.

Analytical approach

Taking the Laplace transform of (5.15) and considering the closed-loop configuration in Figure 5.7, the characteristic function is obtained as the following quasi-polynomial:

$$\Delta(s; \mathbf{k}, h, \tau) = sQ(s) + \left(k_i + s(k_p + k_\delta e^{-hs})\right)P(s)e^{-\tau s}. \quad (5.17)$$

In this section, the stability regions in the (k_p, k_i, k_δ) parameter space are characterized by identifying the boundaries at which characteristic roots cross the imaginary axis.

The following definitions play a fundamental role in the development of the main results.

DEFINITION

5.3

The *frequency crossing set* $\Omega \subset \mathbb{R}_+ \cup \{0\}$ is defined as the set of all frequencies ω for which there exists at least one triplet (k_p, k_i, k_δ) , with fixed delay h , such that the characteristic equation (5.17) possesses a root at $s = i\omega$.

Moreover, due to the conjugate symmetry of the roots of quasi-polynomials with real coefficients, the analysis is restricted to nonnegative frequencies.

Stability crossing curves

The boundary $\mathcal{T}$ of the stability region in the parameter space is composed of the *real root boundary* (RRB), denoted by $\mathcal{T}_0$, and the *complex root boundary* (CRB), denoted by $\mathcal{T}_\omega$, that is,

$$\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_\omega.$$

The following propositions provide explicit expressions for the construction of these hypersurfaces.

PROPOSITION

5.3

Let $h, \tau \in \mathbb{R}_+$ be fixed delays. Then, the real root boundary $\mathcal{T}_0 \subset \mathbb{R}^3$ is defined as the set of all parameters (k_p, k_i, k_δ) for which the characteristic equation (5.17) has a root at $s = 0$. This set is given by the plane

$$\mathcal{T}_0 := \left\{ (k_p, k_i, k_\delta) \in \mathbb{R}^3 \mid k_i = 0, \quad k_p = -\frac{Q(0)}{P(0)} - k_\delta \right\}, \quad (5.18)$$

where $P(s)$ and $Q(s)$ are given in (5.14). The set $\mathcal{T}_0$ defines a stability crossing curve corresponding to a crossing through the origin of the complex plane.

Proof. Substituting $s = 0$ into (5.17) yields $0 \cdot Q(0) + (k_i + 0 \cdot (k_p + k_\delta))P(0) \cdot 1 = k_i P(0) = 0$. Since $P(0) \neq 0$ by Assumption 5.1-(iii), we obtain $k_i = 0$. The condition $k_p + k_\delta = -Q(0)/P(0)$ follows from the steady-state requirement $\lim_{s \rightarrow 0} s^{-1} \Delta(s) = 0$, which ensures the integral action eliminates the steady-state error. Hence, $\mathcal{T}_0$ is exactly the plane described in (5.18). $\square$

PROPOSITION

5.4

Let $h, \tau \in \mathbb{R}_+$ be fixed delays. Assume that $\omega \in \Omega$ satisfies $\omega \neq n\pi/h$ for $n \in \mathbb{N}$. Then, the complex root boundary $\mathcal{T}_\omega \subset \mathbb{R}^3$ is defined as the set of all parameters (k_p, k_i, k_δ)

for which the characteristic equation (5.17) has a pair of purely imaginary roots $s = \pm i\omega$. This set is given by

$$\mathcal{T}_\omega := \bigcup_{\omega \in \Omega} \left\{ \mathbf{k} \in \mathbb{R}^3 \left| \begin{array}{l} k_i = \mathcal{R}(\omega) - \omega \sin(h\omega) k_\delta, \\ k_p = \frac{\mathcal{I}(\omega)}{\omega} - \cos(h\omega) k_\delta, \end{array} \right. k_\delta \in \mathbb{R} \right\},$$

where the frequency-dependent functions $\mathcal{R}(\omega)$ and $\mathcal{I}(\omega)$ are defined as

$$\begin{aligned} \mathcal{R}(\omega) &:= \Re \left\{ -i\omega \frac{Q(i\omega)}{P(i\omega)} e^{i\tau\omega} \right\}, \\ \mathcal{I}(\omega) &:= \Im \left\{ -i\omega \frac{Q(i\omega)}{P(i\omega)} e^{i\tau\omega} \right\}. \end{aligned}$$

Proof. Setting $s = i\omega$ in (5.17) and multiplying by $e^{i\tau\omega}$ (which is nonzero) gives

$$k_i + i\omega k_p + i\omega k_\delta e^{-ih\omega} = -i\omega \frac{Q(i\omega)}{P(i\omega)} e^{i\tau\omega}.$$

Define $H(i\omega) := -i\omega \frac{Q(i\omega)}{P(i\omega)} e^{i\tau\omega} = \mathcal{R}(\omega) + i\mathcal{I}(\omega)$. Using $e^{-ih\omega} = \cos(h\omega) - i\sin(h\omega)$ and equating real and imaginary parts yields the linear system

$$\begin{cases} k_i + \omega \sin(h\omega) k_\delta = \mathcal{R}(\omega), \\ \omega k_p + \omega \cos(h\omega) k_\delta = \mathcal{I}(\omega). \end{cases}$$

Solving for k_i and k_p directly leads to the parametric description of $\mathcal{T}_\omega$ given above. The union over all $\omega \in \Omega$ collects all possible crossing frequencies. $\square$

Remark 5.8. For each fixed value of ω , the set $\mathcal{T}_\omega$ defines a line in the (k_p, k_i) -plane parameterized by k_δ . Consequently, the complete complex root boundary is obtained as a ruled surface in $\mathbb{R}^3$ generated by sweeping ω over the frequency crossing set Ω .

This geometric interpretation plays a fundamental role in the subsequent stability analysis. $\lrcorner$

Remark 5.9. The complete stability crossing set is given by

$$\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_\omega.$$

Moreover, the number of unstable characteristic roots remains invariant within each connected component of $\mathbb{R}^3 \setminus \mathcal{T}$. This property constitutes the basis for determining the stability regions in the controller parameter space. $\lrcorner$

σ –stability

To impose an additional degree of exponential stability, the variable transformation

$$s \rightarrow s - \sigma$$

is introduced, leading to the notion of σ -stability. This reformulation shifts the stability analysis to a translated complex plane, enabling the determination of controller parameter sets for which all characteristic roots of the closed-loop system lie to the left of the vertical line

$$\Re(s) = -\sigma.$$

To analyze the σ -stability of the system, it is necessary to study the characteristic roots located on the shifted imaginary axis, namely

$$s = -\sigma \pm i\omega,$$

for a prescribed controller triplet (k_p, k_i, k_δ) .

In this context, let $\mathcal{T}_\sigma$ denote the set of controller parameters $\mathbf{k}$ for which the quasi-polynomial (5.17) possesses at least one root on the vertical line

$$\mathbf{L}_\sigma := \{s \in \mathbb{C} \mid s = -\sigma + i\omega, \omega \in \mathbb{R}\}.$$

Throughout the remainder of this section, $\mathbf{L}_\sigma$ will be referred to as the σ -axis.

PROPOSITION
5.5

The stability crossing boundaries associated to (5.15) are described with $\sigma \in \mathbb{R}_+$ defined in (5.19)

$$\begin{aligned} k_i(\omega, \sigma, k_\delta, h) &:= \frac{1}{\omega} \left[\sigma \Im\{(\sigma - i\omega) G(i\omega - \sigma)^{-1}\} \right. \\ &\quad \left. + \omega \Re\{(\sigma - i\omega) G(i\omega - \sigma)^{-1}\} \right. \\ &\quad \left. - e^{\sigma h} (\sigma^2 + \omega^2) \sin(\omega h) k_\delta \right], \end{aligned} \quad (5.19)$$

$$\begin{aligned} k_p(\omega, \sigma, k_\delta, h) &:= \frac{1}{\omega} \left[\Im\{(\sigma - i\omega) G(i\omega - \sigma)^{-1}\} \right. \\ &\quad \left. - e^{\sigma h} (\omega \cos(\omega h) + \sigma \sin(\omega h)) k_\delta \right], \end{aligned}$$

con $\omega = 0$ defined in (5.20)

$$\begin{aligned} k_i(\sigma, k_p, k_\delta) &:= \sigma \left[G(-\sigma)^{-1} e^{-\tau\sigma} + k_p + k_\delta e^{h\sigma} \right], \\ k_p &\in \mathbb{R}, \end{aligned} \quad (5.20)$$

where the term $G(\cdot)^{-1}$ is defined by $G(\xi)^{-1} = \frac{Q(\xi)}{P(\xi)} e^{\tau\xi}$.

Proof. The proof follow the same steps in Propositions 5.3-5.4 under the change of variable $s = -\sigma + i\omega$. $\square$

Numerical examples

In this section, the applicability and performance of the proposed I-P δ controller are illustrated through three representative case studies. These examples include a practical first-order motor model, an unstable non-minimum phase system, and a high-order plant, thereby demonstrating the generality of the stability boundaries derived in Section 5.2.

For all case studies, the stability regions are generated considering two representative delay values, namely $h_1 = 0.01$ s and $h_2 = 0.5$ s. The closed-loop performance is evaluated using unit-step reference signals together with constant disturbances $d(t)$ of magnitude ± 0.5 , applied at $t_1 = 40$ s and $t_2 = 80$ s, respectively.

Example 5.3: DC Motor Velocity Control (First-Order)

Consider the following first-order transfer function, which represents the relationship between the input voltage and the angular velocity of a DC motor [68]:

$$G(s) = \frac{1.0646e^{-0.084s}}{0.056s + 1}. \quad (5.21)$$

The corresponding stability regions are shown in Figures. 5.9 and 5.10, a comparison between figures reveals that increasing the delay h significantly modifies the geometry of the stability surface. In particular, as the delay increases from $h = 0.01$ s to $h = 0.5$ s, the stability region shrinks considerably, making the controller tuning process more restrictive.

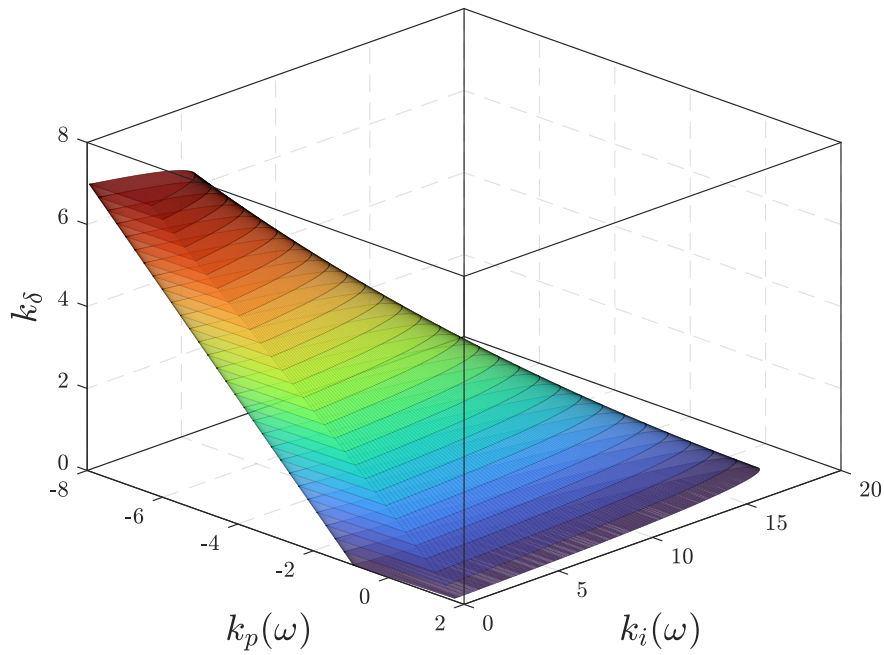


Figure 5.9 / Stability regions (k_i, k_p, k_δ) for the first-order system (5.21) with $h = 0.01$ s.

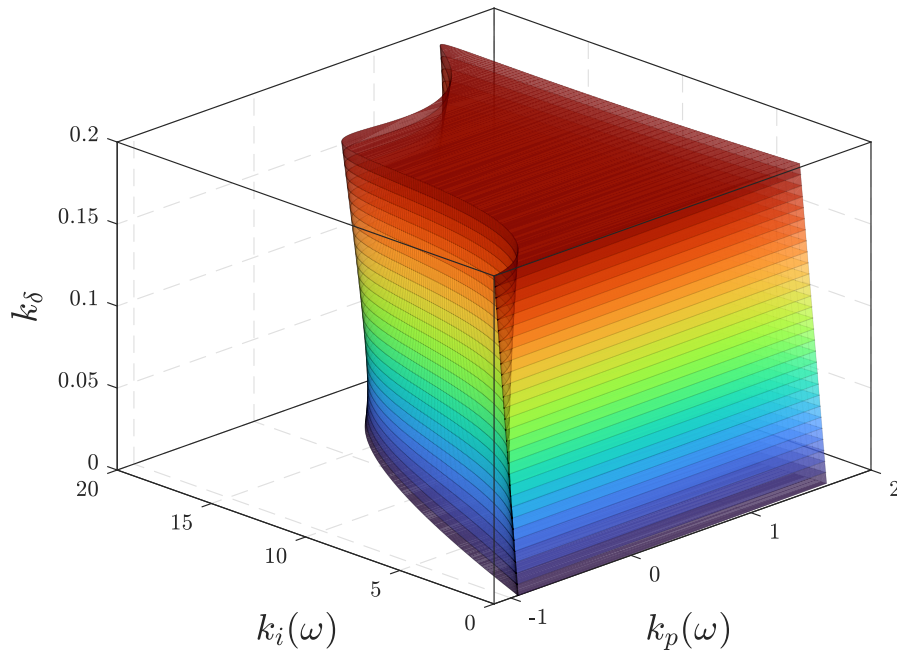


Figure 5.10 / Stability regions (k_i, k_p, k_δ) for the first-order system (5.21) with $h = 0.5$ s.

A controller with parameters

$$(k_p, k_i, k_\delta, h) = (-0.9221, 0.5499, 0.4746, 0.01)$$

is selected from the stability region. The corresponding closed-loop response is presented in Figure 5.11, where satisfactory reference tracking and disturbance rejection capabilities can be observed.

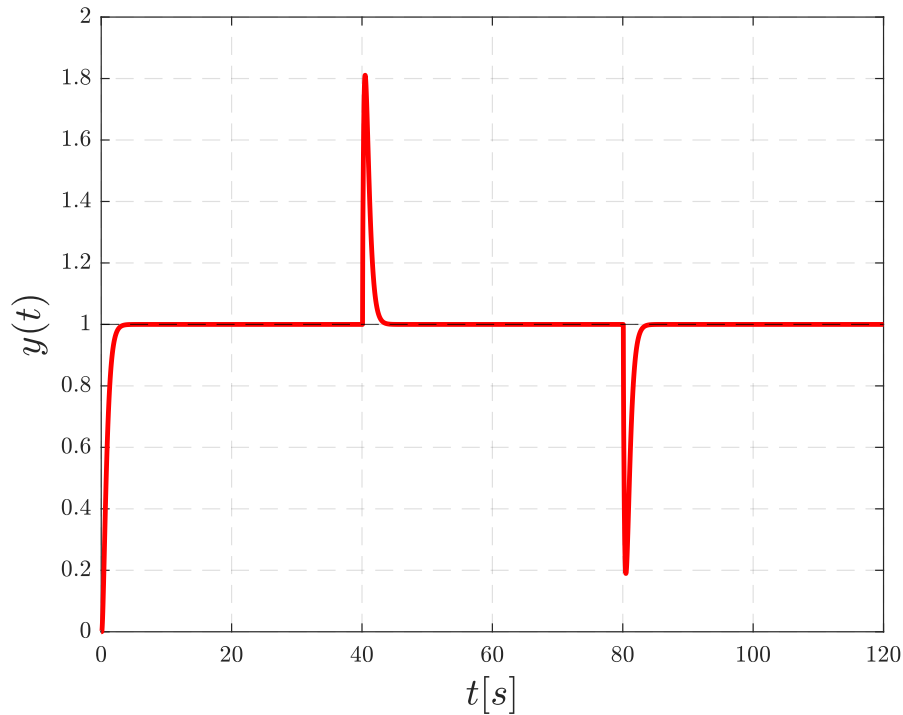


Figure 5.11 / Time response for I-P δ controller with $h = 0.01$ s.

Example 5.4: Unstable Non-Minimum Phase System

Consider the following second-order non-minimum phase unstable plant, taken from [89] [90]:

$$G(s) = \frac{s-2}{s^2 - \frac{1}{2}s + \frac{13}{4}} e^{-2s}. \quad (5.22)$$

The open-loop system is unstable, with poles located at $s_{1,2} = 0.25 \pm i 1.785$. The corresponding stability regions are shown in Figures 5.12-5.13.

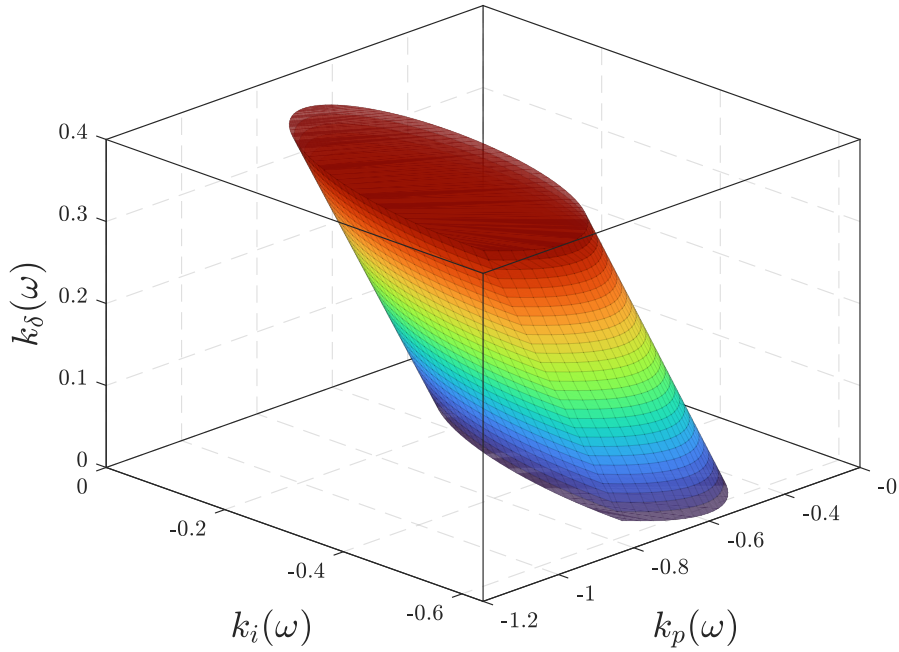


Figure 5.12 / Stability regions (k_i, k_p, k_δ) for the second-order system (5.22) with $h = 0.01$ s.

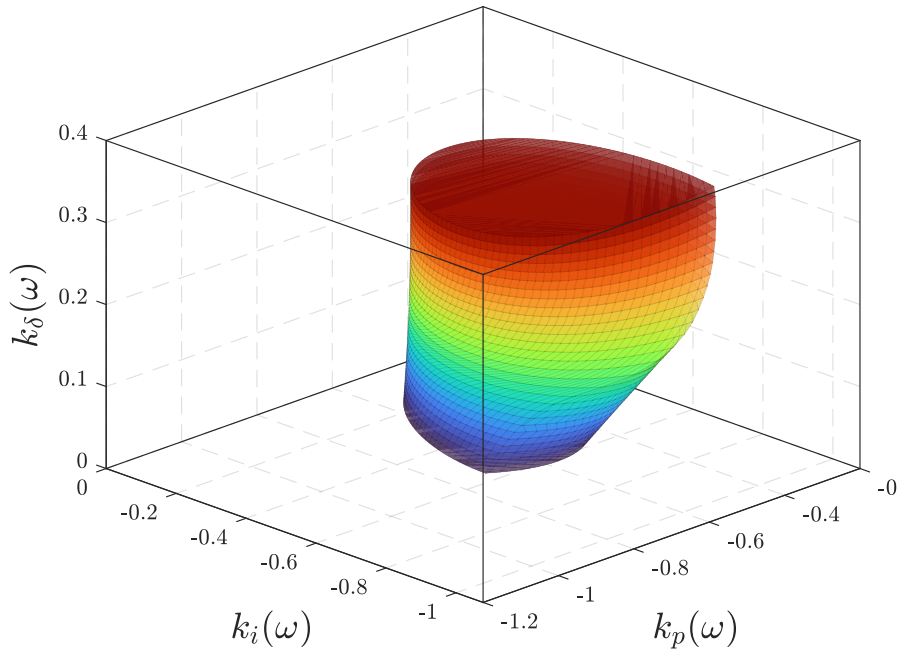


Figure 5.13 / Stability regions (k_i, k_p, k_δ) for the second-order system (5.22) with $h = 0.5$ s.

To improve the exponential convergence rate of the closed-loop response, Proposition 5.5 is now considered. Its application is illustrated through two different

scenarios, beginning with the case where the delay is fixed at $h = 0.01$ s.

As observed from the corresponding figures, the geometry of the stability region changes significantly when the σ -stability condition is imposed.

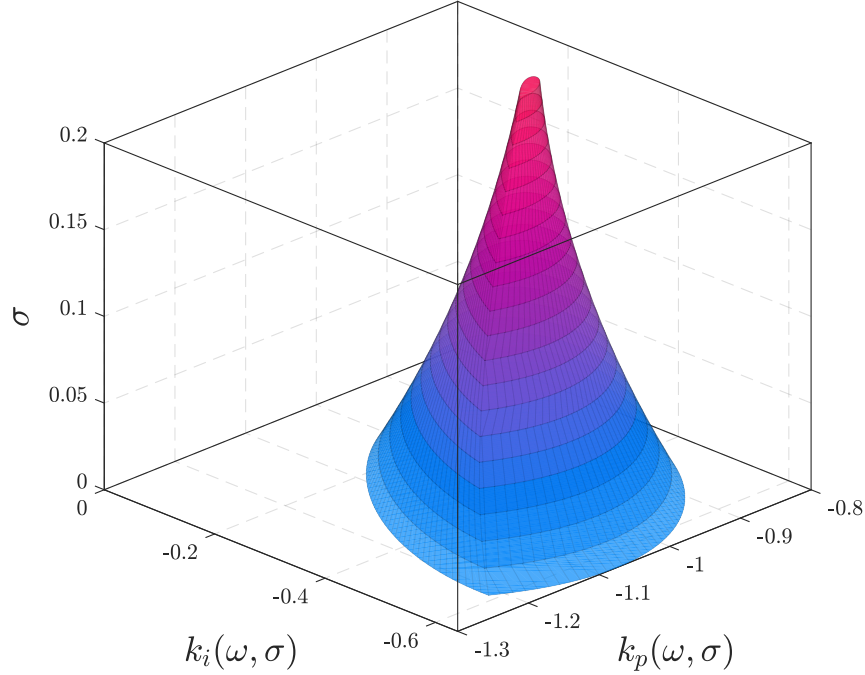


Figure 5.14 / (k_p, k_i) -plane for $h = 0.01$ s and $k_\delta = 0.5$ with σ -stability with $h = 0.01$ s.

Observe that, as σ increases, the corresponding stability region progressively shrinks. Consequently, selecting controllers within the 3D region shown in Figure 5.14 leads to closed-loop responses with improved decay rates.

From a spectral viewpoint, this behavior implies that the dominant characteristic roots are shifted further to the left in the complex plane as σ increases. As a direct consequence, the transient response becomes faster, since the exponential convergence rate of the system is determined by the location of these dominant roots.

To illustrate the closed-loop performance, two representative controller configurations are selected, each corresponding to a different σ -stability region:

(1) The controller

$$(k_p, k_i, k_\delta, h) = (-0.58, -0.25, 0.15, 0.01),$$

whose response is shown in Figure 5.15.

(2) The controller

$$(k_p, k_i, k_\delta, h) = (-0.75, -0.75, 0.35, 0.5),$$

whose response is presented in Figure 5.16.

As expected, the larger delay term ($h = 0.5$ s) introduces a stronger damping effect on the oscillatory behavior of the closed-loop system when compared with the quasi-derivative case ($h = 0.01$ s).

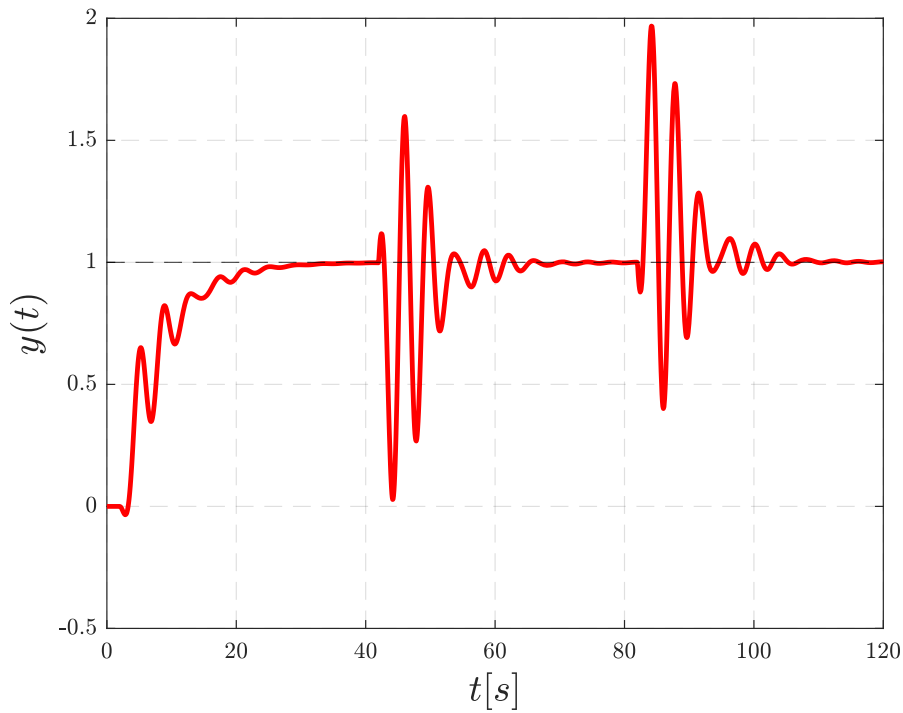


Figure 5.15 / Time response for I-P δ controller with $h = 0.01$ s.

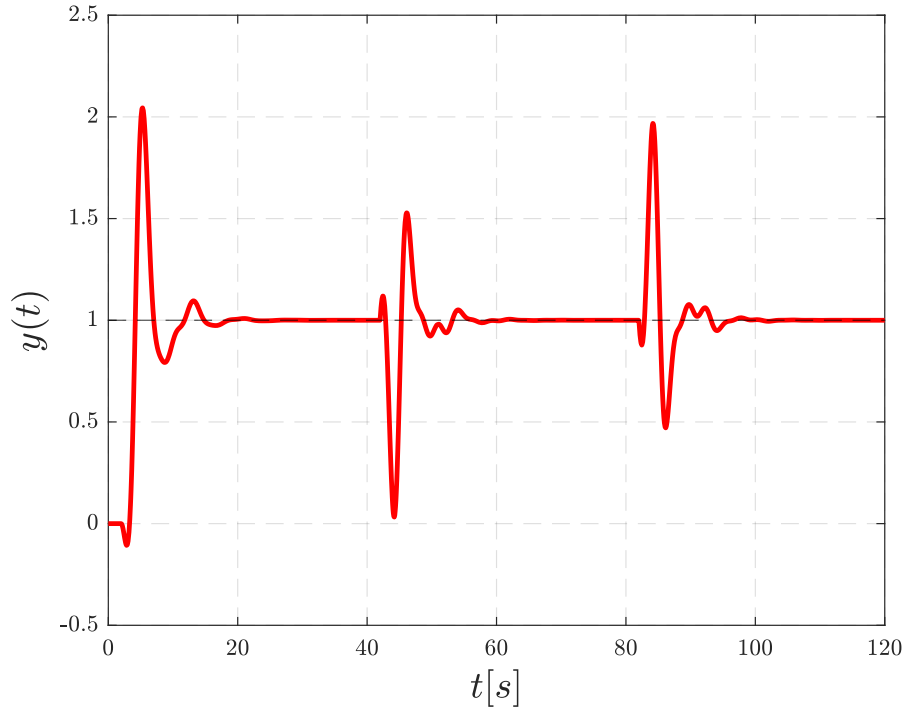


Figure 5.16 / Time response for I-P δ controller with $h = 0.5$ s.

As this process continues, a limiting value is eventually reached, commonly referred to as the maximum σ . This value corresponds to the largest real value of σ for which the stability region collapses into a single point.

At this stage, the dominant characteristic roots cannot be shifted further to the left in the complex plane, implying that no additional improvement in the exponential decay rate can be achieved within the proposed controller structure. Consequently, this point represents the best attainable convergence rate of the closed-loop system under the considered control law.

Example 5.5: High-Order Process (Sixth-Order)

Finally, a sixth-order plant is considered in order to demonstrate the scalability of the proposed methodology [93]:

$$G(s) = \frac{-s^4 - 7s^3 - 2s + 1}{(s+1)(s+2)(s+3)(s+4)(s^2+s+1)} e^{-\frac{1}{20}s}. \quad (5.23)$$

The corresponding stability regions are depicted in Figures 5.17-5.18. For the case $h = 0.01$ s (Figure 5.17), the stability surface exhibits a nearly planar and regular geometry. In contrast, when the delay is increased to $h = 0.5$ s (Figure 5.18), the stability region becomes significantly more distorted, particularly along the k_i direction, directly affecting the distribution of feasible

controller gains.

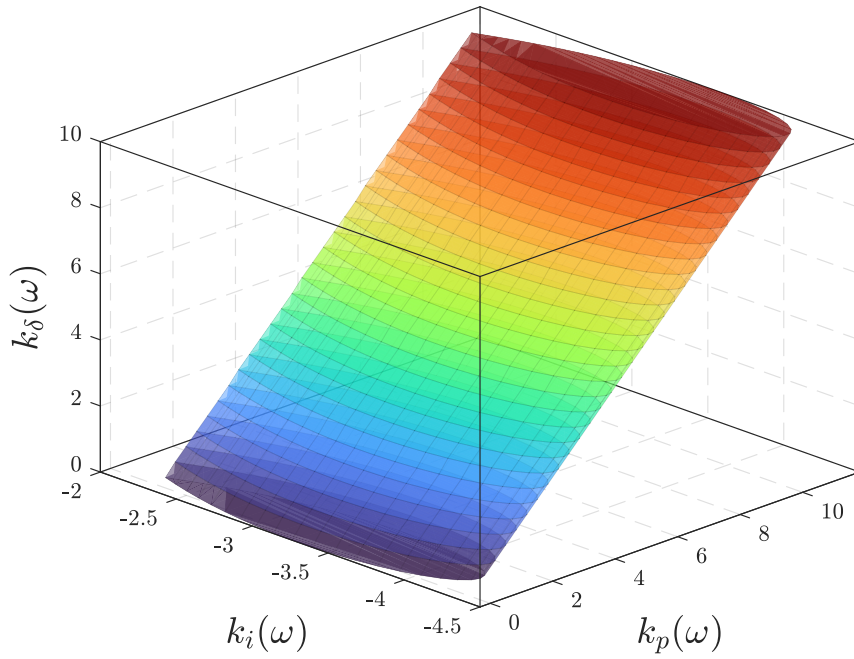


Figure 5.17 / Stability regions (k_i, k_p, k_δ) for the second-order system (??) with $h = 0.01$ s.

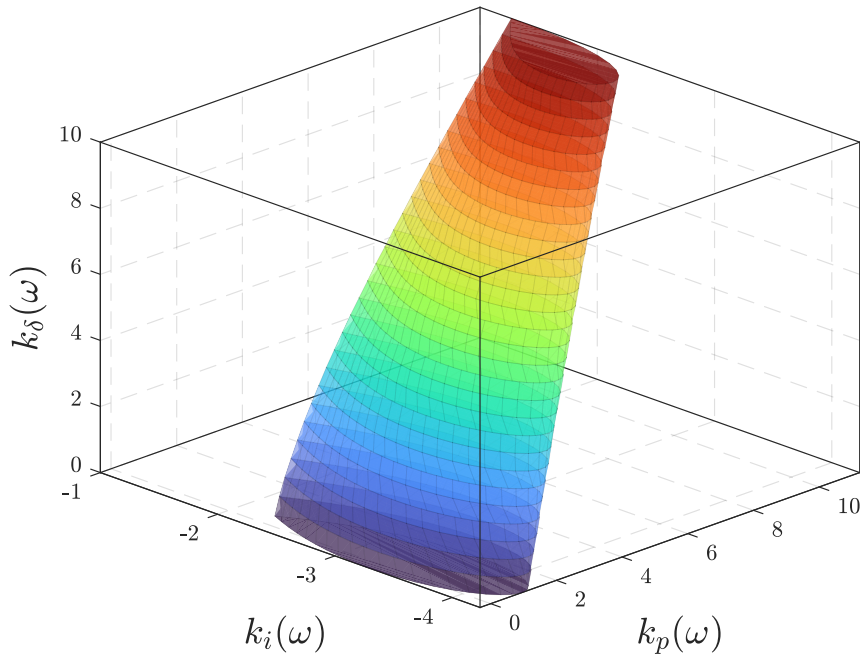


Figure 5.18 / Stability regions (k_i, k_p, k_δ) for the second-order system (??) with $h = 0.5$ s.

6-Level Control of an Underwater Drone

Science never solves one problem without creating ten others.

George Bernard Shaw

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This work was carried out in collaboration with the *Universidad Politécnica de Madrid* (UPM), particularly with the *Escuela Técnica Superior de Ingeniería y Diseño Industrial* (ETSIDI) and the *Centro de Automática y Robótica* (CAR), together with *Monodon*, the research and development division of *Navantia* focused on high-technology projects.

The work developed in this thesis focuses on the trajectory-tracking control of the end-effector of a parallel mechanism. The remaining aspects of the project were developed by other Ph.D. candidates from the aforementioned institution.

Special thanks are extended to *Ph.D. Cecilia Garcia Cena* for accepting to collaborate with us. The author also gratefully acknowledges *M.Eng. Eliseo Cortés Torres*, *M.Eng. Luis García Gonzales*, and *M.Eng. Luis Villamizar Marin* for all the knowledge and experience they shared throughout this work.

The kinematic and dynamic model presented in this section is studied in greater detail in the works [94] and [95], therefore, the kinematic dynamic model itself does not constitute a contribution of this work; rather, the main contribution lies in the development and implementation of the closed-loop control strategy.

Automatic control in underwater exploration is essential to guarantee the safety and efficiency of missions in aquatic environments. In this context, control refers to the ability of an underwater robotic vehicle to maintain its desired orientation, velocity, and depth during operation. To achieve this objective, a variety of control systems, devices, and strategies are employed to ensure that the vehicle follows the planned trajectory, even in complex and difficult-to-access environments.

The need for control in underwater robotics becomes evident when considering

the risks and challenges associated with the underwater environment. Navigation beneath the water surface involves dealing with ocean currents, varying pressure conditions, natural obstacles, and reduced visibility. Any deviation from the desired trajectory or instability in the vehicle motion may lead to mission failure, structural damage, or even the loss of the robotic platform, particularly during deep-sea exploration or operations near sensitive structures.

For these reasons, it is crucial to develop highly accurate and reliable control systems capable of guaranteeing both the integrity of the underwater vehicle and the successful execution of the mission.

6.1 Kinematic Analysis

Figure 6.1b shows the arrangement of the hybrid spatial mechanism with three active degrees of freedom (DoF). It also includes the local reference frames assigned to each joint.

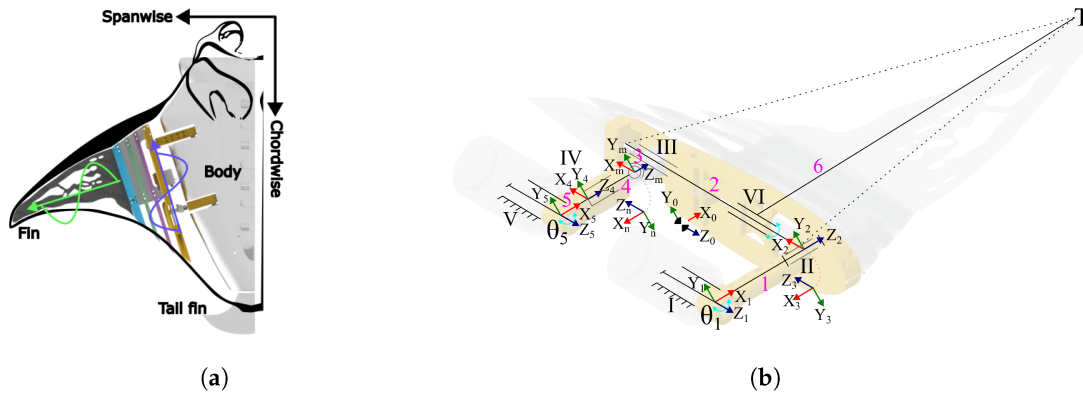


Figure 6.1 / The 3-DoF robotic fin and its corresponding kinematic representation: (a) coupled hybrid mechanism superimposed on an illustrative image of a manta ray (*Mobula alfredi*), and (b) three-dimensional representation of the mechanism, in which joints I, V, and VI are actively driven, whereas joints II, III, and IV remain passive [94].

Using a Lie algebra-based methodology, the complete spatial parallel mechanism (SPM) is divided into two subsystems. The first subsystem corresponds to the spatial mechanism, extending from S_0 to S_6 , and possesses two degrees of freedom. The second subsystem consists of a serial mechanism with one degree of freedom, described from S_6 to S_{Tf} . In the following subsections, each subsystem is modeled mathematically [94] [95].

Forward Kinematics of the Two-Degree-of-Freedom Spatial Mechanism

The method employed is based on Lie algebra, as described in [62]. An i -th coordinate frame is assigned to each rigid body in order to describe its position

with respect to a global reference frame $\{S_0\}$. This strategy transforms closed-loop mechanisms into open-loop mechanisms constrained by position and orientation at a common joint.

Two vectors are defined: one associated with the actuated joints (6.1) and another associated with the passive joints (6.2),

$$\boldsymbol{\theta}_{\text{act}} = [\theta_5, \theta_1] \in \mathbb{R}^2, \quad (6.1)$$

$$\boldsymbol{\theta}_{\text{pas}} = [d_{z4}, \theta_2, d_{z3}] \in \mathbb{R}^3. \quad (6.2)$$

where θ_i and d_{zi} denote the variables associated with revolute and prismatic joints, respectively.

To obtain an open-loop representation of the mechanism, the proposed procedure is applied to the spherical joint located where the coordinate frames $\{S_m\}$ and $\{S_n\}$ meet, as shown in Figure 6.1b. This separates the spherical joint into two kinematic chains. The first chain goes from the frame $\{x_0, y_0, z_0\}$ to $\{x_m, y_m, z_m\}$, while the second goes from $\{x_0, y_0, z_0\}$ to $\{x_n, y_n, z_n\}$.

Kinematic chain #1: The *homogeneous transformation matrix (HTM)* belonging to SE(3) between the frames $\{x_0, y_0, z_0\}$ and $\{x_5, y_5, z_5\}$ is defined as

$$\mathbf{T}_{0,5} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_{CM} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(\theta_5) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4 \times 4}, \quad (6.3)$$

where $\mathbf{r}_{CM} \in \mathbb{R}^{3 \times 1}$ is the *position vector between the global center of mass (CoM)* and the frames S_1 and S_5 .

For the frames $\{x_5, y_5, z_5\}$ and $\{x_4, y_4, z_4\}$, the HTM is defined as

$$\mathbf{T}_{5,4} = \begin{bmatrix} \mathbf{R}_y(\pi/2) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_z(l_5 + d_{z4}) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}, \quad (6.4)$$

where $\mathbf{r}_z$ denotes the position vector associated with the cylindrical joint in $\{S_{4,5}\}$.

Finally, for the frames $\{x_4, y_4, z_4\}$ and $\{x_m, y_m, z_m\}$, the HTM is given by

$$\mathbf{T}_{4,m} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_z(l_4) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (6.5)$$

Kinematic chain #2: The second kinematic chain connects the reference frames from $\{x_0, y_0, z_0\}$ to $\{x_n, y_n, z_n\}$. For the first transformation, defined between $\{x_0, y_0, z_0\}$ and $\{x_1, y_1, z_1\}$, the corresponding homogeneous transformation matrix can be written as

$$\mathbf{T}_{0,1} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_{CM} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(\theta_1) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (6.6)$$

The transformations between the reference frames $\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$, and between $\{x_2, y_2, z_2\}$ and $\{x_3, y_3, z_3\}$, are defined by (6.7) and (6.8), respectively:

$$\mathbf{T}_{1,2} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_x(l_1) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_y(\pi/2) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_z(\theta_2) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}, \quad (6.7)$$

where $\mathbf{r}_x$ represents the position vector from joint 1 to joint 2, expressed in the reference frame $\{S_{1,2}\}$.

For the transformation between frames $\{x_2, y_2, z_2\}$ and $\{x_3, y_3, z_3\}$, one has

$$\mathbf{T}_{2,3} = \begin{bmatrix} \mathbf{R}_y(\pi/2) & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_z(l_2 + d_{z3}) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (6.8)$$

Finally, the transformation between $\{x_3, y_3, z_3\}$ and $\{x_n, y_n, z_n\}$ is given by

$$\mathbf{T}_{3,n} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{r}_x(l_3) \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix}. \quad (6.9)$$

The mechanism described by the previous kinematic model is illustrated in Figure 6.2.

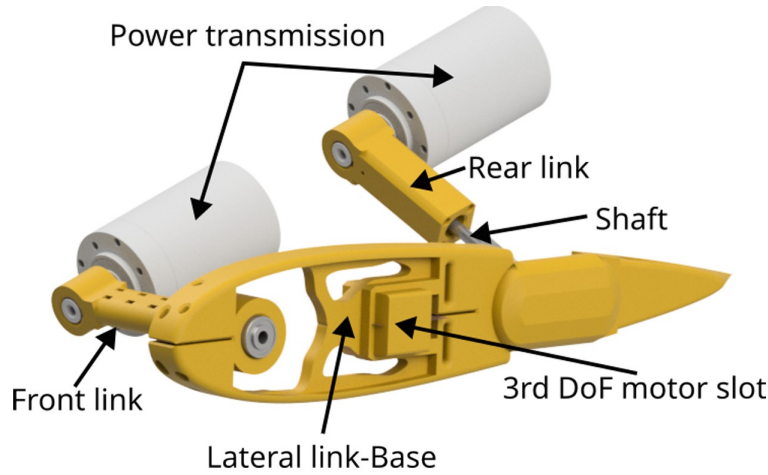


Figure 6.2 / Overview of the proposed 3D arrangement for the 2-DoF spatial parallel mechanism (SPM) [94].

Constraint vector

The motion of the mechanism is defined from the position analysis of the spherical joint together with the physical constraints introduced at the decoupled joint. A spherical joint provides three DoFs, corresponding to rotations about the three coordinate axes.

The constraints correspond to the linear displacements along the three coordinate axes and can be expressed as follows

$$\Phi = \begin{bmatrix} \mathbf{R}_{0,n} \mathbf{u}_{x,x} \cdot (\mathbf{r}_n - \mathbf{r}_m) \\ \mathbf{R}_{0,n} \mathbf{u}_{y,y} \cdot (\mathbf{r}_n - \mathbf{r}_m) \\ \mathbf{R}_{0,n} \mathbf{u}_{z,z} \cdot (\mathbf{r}_n - \mathbf{r}_m) \end{bmatrix}, \quad (6.10)$$

where $\mathbf{u}_{x,x}$, $\mathbf{u}_{y,y}$, $\mathbf{u}_{z,z}$ are unit vectors associated with the local axes of each joint, while $(\mathbf{r}_n - \mathbf{r}_m)$ represents the position difference between the reference frames $\{x_n, y_n, z_n\}$ and $\{x_m, y_m, z_m\}$, expressed with respect to the base reference frame.

Jacobian of the constraint vector

The time derivative of the *constraint vector* can be written as $\dot{\Phi} = \Phi_\theta \dot{\theta}_i$, where $\dot{\theta}_i$ contains the joint velocities and Φ_θ denotes the Jacobian matrix associated with the constraints. Additional details can be found in [62] [94] [95]. According to the time derivative of the constraint vector, $\dot{\theta}_i$ can be represented in terms of Lie algebra as

$$\Phi_\theta(j, i) = \varepsilon_{j,i} \mathbf{C}_{nj,nj}^\top \text{Ad}_{\mathbf{T}_{nj,i}} \mathcal{S}_i. \quad (6.11)$$

The indices i and j correspond to the j -th constraint and the joint degree of freedom θ_i , respectively. Here, $\mathcal{S}_i \in \mathfrak{se}(3)$ represents the velocity associated with the i -th joint and is defined according to its corresponding degree of freedom θ_i .

The scalar $\varepsilon_{j,i}$ is defined as

$$\varepsilon_{j,i} = \begin{cases} 1, & \text{if } \theta_i \text{ belongs to the right side of the kinematic loop involved} \\ & \text{in constraint } j, \\ 0, & \text{if } \theta_i \text{ is not required to evaluate constraint } j, \\ -1, & \text{if } \theta_i \text{ belongs to the left side of the kinematic loop involved} \\ & \text{in constraint } j. \end{cases}$$

The covector $\mathbf{C}_{nj,nj} \in \mathbb{R}^{6 \times 1}$ corresponds to the j -th constraint and describes the reaction wrench generated at the decoupled joint. Moreover, $\text{Ad}_{\mathbf{T}_{nj,i}} \in \mathbb{R}^{6 \times 6}$ denotes the adjoint matrix of $\text{SE}(3)$, which maps the vector $\mathcal{S}_i$ from joint i to the reference frame fixed to body n .

In general, each term $\mathbf{C}_{nj,nj}^\top \text{Ad}_{\mathbf{T}_{nj,i}} \mathcal{S}_i$ expresses how the j -th constraint is related to the degree of freedom of joint i , whereas $\dot{\theta}_i$ contains the corresponding joint velocities.

Thus, the twists associated with the Jacobian matrix Φ_θ for each active and passive joint can be expressed as

$$\mathcal{S}_{\text{rot}} = [0, 0, 1, 0, 0, 0]^\top \in \mathbb{R}^{6 \times 1}, \quad \mathcal{S}_{\text{tras}} = [0, 0, 0, 0, 0, 1]^\top, \quad (6.12)$$

where $\mathcal{S}_{\text{rot}}$ represents the twist associated with revolute joints, namely the axes $\{S_2\}$, $\{S_4\}$, and $\{S_5\}$, while $\mathcal{S}_{\text{tras}}$ represents the twist associated with prismatic joints, namely $\{S_3\}$ and $\{S_6\}$.

Finally, for the bodies belonging to the left side of the kinematic loop, the adjoint operator $\text{Ad}_{\mathbf{T}_{nj,i}}$ is decomposed into the product of two adjoint operators:

$$\text{Ad}_{\mathbf{T}_{nj,i}} = \text{Ad}_{\mathbf{T}_{nj,mj}} \text{Ad}_{\mathbf{T}_{mj,i}},$$

where $\text{Ad}_{\mathbf{T}_{mj,i}}$ is the adjoint operator obtained from the same open kinematic chain located on the left side, and $\text{Ad}_{\mathbf{T}_{nj,mj}}$ is the adjoint operator computed from the constraints and the homogeneous transformation matrices associated with bodies mj and nj , namely, $\mathbf{T}_{nj,mj} = \mathbf{T}_{nj,0} \mathbf{T}_{0,mj}$.

For the system studied in Figure 6.2, the previous formulation can be summarized as

$$\Phi_{\theta} = \begin{bmatrix} -\mathbf{C}_x^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,2}} S_2 & -\mathbf{C}_x^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,3}} S_3 & \mathbf{C}_x^{\top} \text{Ad}_{\mathbf{T}_{n,4}} S_4 & \mathbf{C}_x^{\top} \text{Ad}_{\mathbf{T}_{n,5}} S_5 & \mathbf{C}_x^{\top} \text{Ad}_{\mathbf{T}_{n,6}} S_6 \\ -\mathbf{C}_y^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,2}} S_2 & -\mathbf{C}_y^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,3}} S_3 & \mathbf{C}_y^{\top} \text{Ad}_{\mathbf{T}_{n,4}} S_4 & \mathbf{C}_y^{\top} \text{Ad}_{\mathbf{T}_{n,5}} S_5 & \mathbf{C}_y^{\top} \text{Ad}_{\mathbf{T}_{n,6}} S_6 \\ -\mathbf{C}_z^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,2}} S_2 & -\mathbf{C}_z^{\top} \text{Ad}_{\mathbf{T}_{n,m}} \text{Ad}_{\mathbf{T}_{m,3}} S_3 & \mathbf{C}_z^{\top} \text{Ad}_{\mathbf{T}_{n,4}} S_4 & \mathbf{C}_z^{\top} \text{Ad}_{\mathbf{T}_{n,5}} S_5 & \mathbf{C}_z^{\top} \text{Ad}_{\mathbf{T}_{n,6}} S_6 \end{bmatrix}, \quad (6.13)$$

where the constraint Jacobian matrix is of dimension $n_c \times n_v$, with n_c representing the number of constraints and n_v the number of degrees of freedom (either active or passive) associated with the joints composing the open kinematic chains.

The constraint Jacobian matrix can be partitioned by considering the active and passive joints as follows:

$$\Phi_{\theta_{\text{act}}} = [\Phi_{\theta}(:, 1), \Phi_{\theta}(:, 3)] \in \mathbb{R}^{3 \times 2}, \quad (6.14)$$

$$\Phi_{\theta_{\text{pas}}} = [\Phi_{\theta}(:, 2), \Phi_{\theta}(:, 5), \Phi_{\theta}(:, 6)] \in \mathbb{R}^{3 \times 3}. \quad (6.15)$$

These constraints are holonomic in nature, since the system is not fully actuated.

Position analysis

Within computational kinematics, the Newton–Raphson method is widely used to numerically adjust the *assembly* of kinematic chains whenever a violation of the constraint vector (6.10) occurs. The Newton–Raphson algorithm is based on solving

$$\Phi_{\theta} \Delta \theta = -\Phi, \quad (6.16)$$

where Φ_{θ} is the Jacobian matrix of the constraint vector given in (6.13).

Equation (6.17) defines the kinematic relation between the active and passive joints:

$$\Phi_{\theta_{\text{act}}} \Delta \theta_{\text{act}} + \Phi_{\theta_{\text{pas}}} \Delta \theta_{\text{pas}} = -\Phi. \quad (6.17)$$

When the active joint parameters are considered as reference inputs for the system, that is, values directly prescribed by the user, no numerical correction is required for them. Therefore, it can be assumed that $\Delta \theta_{\text{act}} = 0$.

Consequently, equation (6.17) can be rewritten as

$$\Phi_{\theta_{\text{pas}}} \Delta \theta_{\text{pas}} = -\Phi. \quad (6.18)$$

The Newton–Raphson method was used to compute a numerical solution of this equation. Once the passive joint positions are obtained, the result can be written as

$$\Delta \theta_{\text{pas}} = -(\Phi_{\theta_{\text{pas}}})^\dagger \Phi, \quad (6.19)$$

where

$$\theta_{\text{pas}} \in \mathbb{R}^{3 \times 1},$$

and the operator $(\cdot)^\dagger$ denotes either the matrix inverse (when the matrix is square and nonsingular) or the pseudoinverse when the matrix has more rows than columns. In the latter case, redundant constraints are automatically eliminated by the pseudoinverse method.

Obtaining a numerical solution is not recommended when the matrix has more columns than rows, since this implies that the mechanism possesses more joint variables than constraints. Physically, this indicates that the mechanism lacks the required number of actuators for fully controlled motion.

The Newton–Raphson algorithm based on equation (6.19) is iteratively repeated until the solution satisfies $\Phi = 0$, or, alternatively, until the norm of the constraint vector is minimized as much as possible.

Velocity analysis

Once the constraint vector Φ has been minimized, the velocity analysis can be performed. The main difficulty lies in determining the velocities of the passive joints, since both $\Phi_{\theta_{\text{pas}}}$ and $\Phi_{\theta_{\text{act}}}$ are already known from the position analysis. Therefore, equation (6.17) can be rewritten as

$$\Phi_{\theta_{\text{act}}} \dot{\theta}_{\text{act}} + \Phi_{\theta_{\text{pas}}} \dot{\theta}_{\text{pas}} = 0. \quad (6.20)$$

Thus, the passive joint velocities are described by

$$\dot{\theta}_{\text{pas}} = -(\Phi_{\theta_{\text{pas}}})^\dagger \Phi_{\theta_{\text{act}}} \dot{\theta}_{\text{act}}. \quad (6.21)$$

Once the corrected positions and velocities have been obtained, the acceleration analysis can then be carried out.

Acceleration analysis

The second time derivative of the constraint vector can be expressed as

$$\ddot{\Phi} = \Phi_{\theta} \ddot{\theta}_i - \gamma, \quad (6.22)$$

where γ , in the context of the k -th constraint (i.e., the k -th decoupled joint point), is defined as

$$\gamma_k = -\mathbf{C}_{n,n}^{*T} \left(\mathbf{A}_{c,nk} - \text{Ad}_{\mathbf{T}_{nk,mk}} \mathbf{A}_{c,mk} + \text{ad}_{\mathbf{V}_{nk}} \text{Ad}_{\mathbf{T}_{nk,mk}} \mathbf{V}_{mk} \right). \quad (6.23)$$

In (6.24), the term $\text{ad}_{\mathbf{V}_{nk}} \text{Ad}_{\mathbf{T}_{nk,mk}} \mathbf{V}_{mk}$ corresponds to the *Coriolis acceleration term*, which results from the interaction between the velocities of bodies n and m .

The vector $\mathbf{A}_{c,nk}$ represents the Coriolis acceleration of body n expressed in the reference frame fixed to that body. Similarly, $\text{Ad}_{\mathbf{T}_{nk,mk}} \mathbf{A}_{c,mk}$ gives the Coriolis acceleration of body m expressed in the same reference frame.

For the system represented in Figure 6.1b, this term is expressed as

$$\gamma = -\mathbf{C}^{*T} \left(\mathbf{A}_{c,n} - \text{Ad}_{\mathbf{T}_{n,m}} \mathbf{A}_{c,m} + \text{ad}_{\mathbf{V}_n} \text{Ad}_{\mathbf{T}_{n,m}} \mathbf{V}_m \right), \quad (6.24)$$

where

$$\mathbf{C}^{*T} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6.25)$$

The next step consists of replacing the accelerations $\mathbf{A}_n$ and $\mathbf{A}_m$ in order to obtain the explicit definition of γ . According to the definition in (2.79),

$$\begin{aligned} \mathbf{A}_n &= \sum_{i=1}^n \text{Ad}_{\mathbf{T}_{q,i}} \mathcal{S}_i \ddot{\theta}_i + \mathbf{A}_{c,i}, \\ \mathbf{A}_m &= \sum_{j=1}^m \text{Ad}_{\mathbf{T}_{q,j}} \mathcal{S}_j \ddot{\theta}_j + \mathbf{A}_{c,j}. \end{aligned}$$

The Coriolis acceleration of body m , observed from reference frame n , is given by

$$\mathbf{A}_{c,m} = \text{Ad}_{\mathbf{T}_{m,2}} \text{ad}_{\mathcal{S}_2} \text{Ad}_{\mathbf{T}_{2,3}} \mathcal{S}_3 \dot{\theta}_2 \dot{d}_{z3}. \quad (6.26)$$

Similarly, the Coriolis acceleration of body n , observed from its reference frame, is expressed as

$$\begin{aligned} \mathbf{A}_{c,n} &= \text{Ad}_{\mathbf{T}_{n,4}} \text{ad}_{\mathcal{S}_4} \text{Ad}_{\mathbf{T}_{4,5}} \mathcal{S}_5 \dot{\theta}_4 \dot{\theta}_5 \\ &\quad + \text{Ad}_{\mathbf{T}_{n,4}} \text{ad}_{\mathcal{S}_4} \text{Ad}_{\mathbf{T}_{4,6}} \mathcal{S}_6 \dot{\theta}_4 \dot{d}_{z6} \\ &\quad + \text{Ad}_{\mathbf{T}_{n,5}} \text{ad}_{\mathcal{S}_5} \text{Ad}_{\mathbf{T}_{5,6}} \mathcal{S}_6 \dot{\theta}_5 \dot{d}_{z6}. \end{aligned} \quad (6.27)$$

The main challenge lies in computing the accelerations of the passive joints, since the information associated with the active joints is already known. Therefore,

$$\ddot{\theta}_{\text{pas}} = \left(\Phi_{\theta_{\text{pas}}} \right)^\dagger \left(\gamma - \Phi_{\theta_{\text{act}}} \ddot{\theta}_{\text{act}} \right). \quad (6.28)$$

The velocities associated with bodies n and m are described by

$$\mathbf{V}_n = \sum_{i=1}^n \text{Ad}_{\mathbf{T}_{n,i}} \mathcal{S}_i \dot{\boldsymbol{\theta}}_i, \quad (6.29)$$

$$\mathbf{V}_m = \sum_{j=1}^m \text{Ad}_{\mathbf{T}_{m,j}} \mathcal{S}_j \dot{\boldsymbol{\theta}}_j, \quad (6.30)$$

where the term $\text{ad}_{\mathbf{V}_i}$ denotes the Lie bracket operator.

6.2 Dynamic Analysis

Inverse dynamics is important for analyzing and controlling the motion of complex mechanical systems. It is used to determine the forces and torques required to produce a given motion from the known kinematic and dynamic behavior of the system. By solving the inverse dynamics problem, researchers can infer the forces required to achieve a desired motion or analyze the forces present in a system during motion [96].

This capability enhances our understanding and facilitates practical applications in fields such as bioinspired robotics and mechanical systems, where detailed knowledge of forces and torques is essential for the development of agile and adaptive robots, biomimetic actuators, and compliant mechanisms [97].

The dynamic model presented in this section is studied in greater detail in the works [94] and [95].

Inverse Dynamics of the 2-DoF Spatial Mechanism

Following the design proposed in [98], an initial prototype for the dynamic analysis of the robotic fin mechanism was developed at *ETSIDI-UPM* in collaboration with the research team *Monodon*. The design of the rear link incorporated a commercial SKF spherical joint, which allows rotational motion of up to 14° about all axes. For the lateral link, the NACA0022 profile from Airfoil Tools was employed.

The standard configuration was modified to increase the range of motion of the spherical joint. This was achieved by using the redundancy between the cylindrical and spherical joints, which also increases the workspace of the mechanism. The resulting spatial mechanism is shown in Figure 6.2.

According to [59], the dynamics of a rigid body with a non-centralized reference frame, which is common in bodies belonging to open kinematic chains, can be characterized through the following formulation:

$$\mathbf{F}_i = \mathbf{M}_i \mathbf{A}_i + \text{ad}_{\mathbf{V}_i}^* \mathbf{M}_i \mathbf{V}_i. \quad (6.31)$$

Here, $\mathbf{F}_i$ represents the sum of all forces acting on the system. The operator $\text{ad}_{\mathbf{V}_i}^*$ denotes the Lie bracket operator for covectors. The six-dimensional vector $\mathbf{V}_i$

corresponds to the velocity of the i -th body and is defined as

$$\mathbf{V}_i = \sum_{k=1}^i \text{Ad}_{\mathbf{T}_{i,k}} \mathcal{S}_k \dot{\boldsymbol{\theta}}_k. \quad (6.32)$$

Similarly, the six-dimensional acceleration vector $\mathbf{A}_i$ is given by

$$\mathbf{A}_i = \sum_{k=1}^i \text{Ad}_{\mathbf{T}_{i,k}} \mathcal{S}_k \ddot{\boldsymbol{\theta}}_k + \text{ad}_{\mathbf{V}_i} \mathcal{S}_i \dot{\boldsymbol{\theta}}_i. \quad (6.33)$$

The term $\mathbf{M}_i$ describes the mass properties of the rigid body through the following matrix:

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{R}_{i,c} \mathbf{I}_c \mathbf{R}_{i,c}^\top - m_i \hat{\mathbf{r}}_{i,c} \hat{\mathbf{r}}_{i,c}^\top & m_i \hat{\mathbf{r}}_{i,c} \\ -m_i \hat{\mathbf{r}}_{i,c}^\top & m_i \mathbf{I}_{3 \times 3} \end{bmatrix}. \quad (6.34)$$

The mass of rigid body i is represented by $m_i \in \mathbb{R}_+$. The matrix $\hat{\mathbf{r}}_{i,c}$ is associated with the position of the center of mass (CoM) of body i relative to its local reference frame, whereas $\mathbf{R}_{i,c}$ describes the orientation of the CoM frame with respect to that same local frame. Moreover, $\mathbf{I}_c$ denotes the inertia matrix of the rigid body expressed in its local coordinates, and $\mathbf{I}_{3 \times 3}$ is the identity matrix.

Considering (6.31) and applying the principle of virtual work¹, the inverse dynamics equation for closed kinematic chains is obtained as (6.35), as extensively discussed in [62]:

$$\boldsymbol{\tau} = \sum_{i=2}^6 \mathbf{J}_i^\top (\mathbf{M}_i \mathbf{A}_i + \text{ad}_{\mathbf{V}_i}^* \mathbf{M}_i \mathbf{V}_i - \mathbf{F}_i). \quad (6.35)$$

- $\mathbf{J}_i \in \mathbb{R}^{6 \times 2}$: spatial Jacobian of body i with respect to the active coordinates.
- $\mathbf{M}_i \in \mathbb{R}^{6 \times 6}$: spatial mass matrix of body i .
- $\mathbf{A}_i \in \mathbb{R}^{6 \times 1}$: spatial acceleration (time derivative of the twist) of body i .
- $\text{ad}_{\mathbf{V}_i}^* \in \mathbb{R}^{6 \times 6}$: coadjoint Lie bracket operator for forces, i.e., the dual operator of the Lie bracket.
- $\mathbf{V}_i \in \mathbb{R}^{6 \times 1}$: spatial velocity (twist) of body i .
- $\mathbf{F}_i \in \mathbb{R}^{6 \times 1}$: external force applied to body i .
- $\boldsymbol{\tau} \in \mathbb{R}^{2 \times 1}$: torques applied at the active joints (the passive joints are unactuated).

The previous expression computes the torques required at the active joints in terms of their positions, velocities, and accelerations, which are assumed to be known through sensors, observers, or estimation methods.

¹The principle of virtual work can be derived from d'Alembert's principle, which in turn follows from Newtonian mechanics or, more generally, from the principle of least action.

In the previous equation, $\text{ad}_{\mathbf{V}_i}^*$ denotes the Lie bracket operator for covectors:

$$\text{ad}_{\mathbf{V}_i}^* = \begin{bmatrix} \hat{\boldsymbol{\omega}}_i & \hat{\mathbf{v}}_i \\ \mathbf{0}_{3 \times 3} & \hat{\boldsymbol{\omega}}_i \end{bmatrix}. \quad (6.36)$$

Jacobian Matrix

In (6.35), the term $\mathbf{J}_i$ represents the Jacobian matrix associated with each body, which relates the velocity of each link to the velocities of the actuated joints. To compute $\mathbf{J}_i$, the algorithm proposed in [62] is employed, which requires the computation of a matrix ρ defined as

$$\rho = \Phi_{\theta_{\text{pas}}}^\dagger \Phi_{\theta_{\text{act}}}. \quad (6.37)$$

Thus, for an actuated joint, the Jacobian matrix $\mathbf{J}_i$ is described as

$$\mathbf{J}_i(:, j) = \mathbf{J}_i(:, j) + \text{Ad}_{\mathbf{T}_{i,j}} \mathcal{S}_j, \quad (6.38)$$

whereas for passive joints, $\mathbf{J}_i$ is given by

$$\mathbf{J}_i = \mathbf{J}_i + \text{Ad}_{\mathbf{T}_{i,j}} \mathcal{S}_j \rho_j. \quad (6.39)$$

Inverse model (Inverse Dynamics): The model defined in (6.35) follows the inverse dynamics structure, that is, the model computes the required joint torques from the prescribed position, velocity, and acceleration reference signals:

$$\boldsymbol{\tau} = f(\boldsymbol{\theta}_d, \dot{\boldsymbol{\theta}}_d, \ddot{\boldsymbol{\theta}}_d).$$

The block diagram representing equation (6.35) is shown in Figure 6.3.

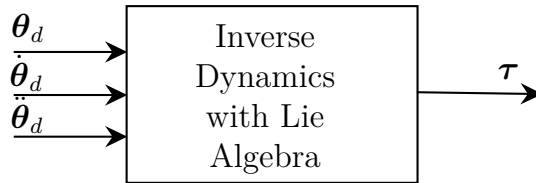


Figure 6.3 / Block diagram of the open-loop inverse model.

The simulation was carried out using the parameters and reference trajectories reported in [94] [95].

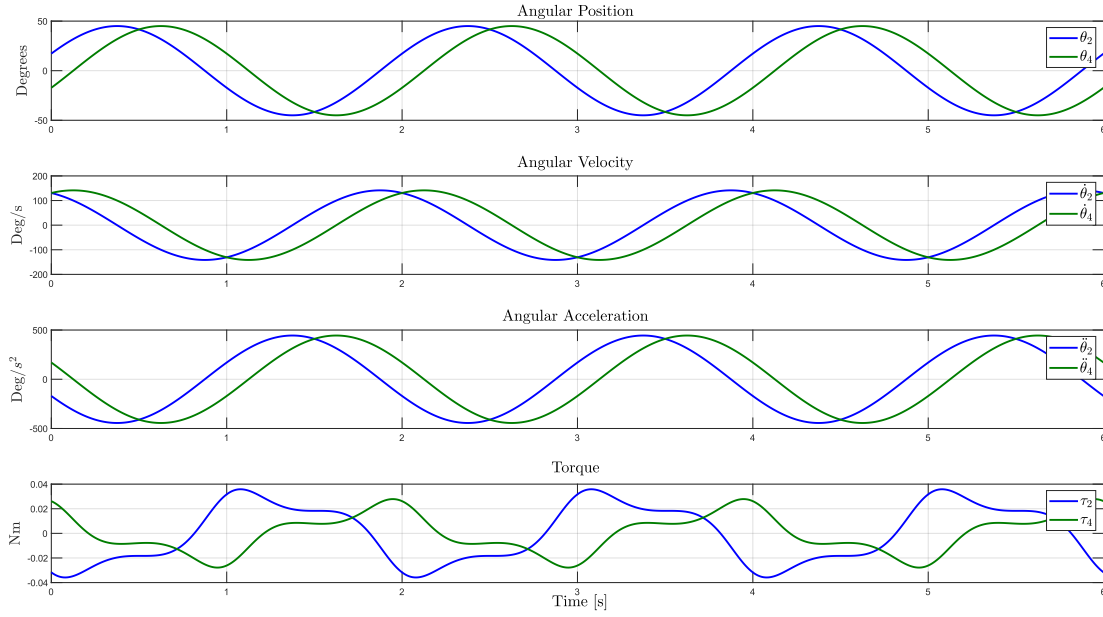


Figure 6.4 / Open-loop responses of the system using the inverse dynamic model.

This model does not allow the direct implementation of a feedback control strategy, since the actuator trajectories are imposed as direct inputs and therefore are not corrected according to the system response. Consequently, it is necessary to obtain a model with a direct dynamics structure.

Forward Dynamics of the 2-DoF Spatial Mechanism

This model allows the computation of accelerations (which can later be integrated to obtain velocities and positions) from the torques directly applied to the actuators, while also allowing the inclusion of external forces:

$$\ddot{\theta}_{\text{act}} = f(\theta_{\text{act}}, \dot{\theta}_{\text{act}}, \theta_{\text{pas}}, \dot{\theta}_{\text{pas}}, \tau).$$

First, the inverse dynamic model is reconsidered:

$$\tau = \sum_{i=2}^6 \mathbf{J}_i^\top (\mathbf{M}_i \mathbf{A}_i + \text{ad}_{\mathbf{V}_i}^* \mathbf{M}_i \mathbf{V}_i - \mathbf{F}_i). \quad (6.40)$$

A dimensional analysis is performed in order to identify how the spatial accelerations of the i -th body can be isolated:

- $\mathbf{J}_i \in \mathbb{R}^{6 \times 2}$: spatial Jacobian of body i with respect to the active coordinates.
- $\mathbf{M}_i \in \mathbb{R}^{6 \times 6}$: spatial mass matrix of body i .
- $\mathbf{A}_i \in \mathbb{R}^{6 \times 1}$: spatial acceleration (time derivative of the twist) of body i .
- $\text{ad}_{\mathbf{V}_i}^* \in \mathbb{R}^{6 \times 6}$: dual gyroscopic operator (dual Lie bracket operator).

- $\mathbf{V}_i \in \mathbb{R}^{6 \times 1}$: spatial velocity (twist) of body i .
- $\mathbf{F}_i \in \mathbb{R}^{6 \times 1}$: external force applied to body i .
- $\boldsymbol{\tau} \in \mathbb{R}^{2 \times 1}$: torque applied at the **active joints** (the passive joints are unactuated).

Considering the contribution of the i -th body torque, one obtains

$$\boldsymbol{\tau}_i = \mathbf{J}_i^\top (\mathbf{M}_i \mathbf{A}_i + \text{ad}_{\mathbf{V}_i}^* \mathbf{M}_i \mathbf{V}_i - \mathbf{F}_i). \quad (6.41)$$

Starting from left matrix multiplication, one has $(\mathbf{J}_i^\top)^{-1} \mathbf{J}_i^\top = 1$ however, it is known that $\mathbf{J}_i^\top$ is not square. Therefore, it becomes necessary to compute its pseudoinverse. After performing the corresponding algebraic manipulations, the following expression is obtained:

$$\mathbf{A}_i = \mathbf{M}_i^{-1} \left(\underbrace{(\mathbf{J}_i^\top)^\dagger \boldsymbol{\tau}}_{\text{forces acting on the body}} - \underbrace{\text{ad}_{\mathbf{V}_i}^* \mathbf{M}_i \mathbf{V}_i}_{\text{Coriolis forces}} + \underbrace{\mathbf{F}_i}_{\text{external forces}} \right), \quad (6.42)$$

where $\mathbf{M}_i^{-1}$ is invertible by definition and $(\mathbf{J}_i^\top)^\dagger$ can always be computed; otherwise, there would exist input directions that do not generate any joint torque.

Grouping the force terms yields

$$\mathbf{F}_{\text{total}} = \mathbf{f}_i - \mathbf{f}_{\text{coriolis}} + \mathbf{F}_i.$$

Thus, equation (6.42) can be summarized as

$$\mathbf{A}_i = \mathbf{M}_i^{-1} \mathbf{F}_{\text{total}}.$$

On the other hand, one of the key aspects in control and simulation using Lie algebra consists in determining how to transform spatial accelerations $\mathbf{A}_i \in \mathbb{R}^6$, which correspond to rigid-body accelerations in $\mathfrak{se}(3)$ (or equivalently a 6D vector), into joint accelerations $\ddot{\boldsymbol{\theta}} \in \mathbb{R}^n$, which can subsequently be integrated, where in this case $n = 5$ joints.

The previous procedure is illustrated in the block diagram of Figure 6.5.

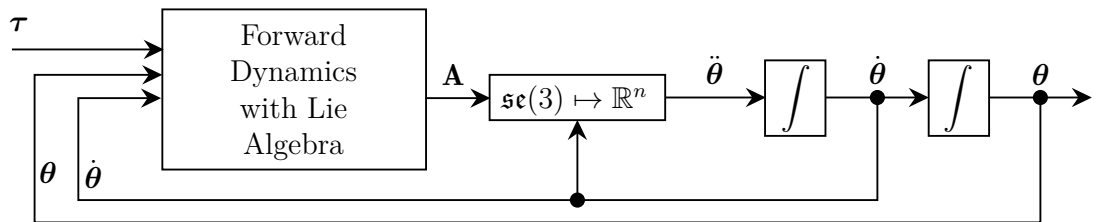


Figure 6.5 / Block diagram of the open-loop direct dynamic model.

What is being performed is not directly a functional mapping $\mathfrak{se}(3) \mapsto \mathbb{R}^n$, but rather the solution of a kinematic compatibility equation in $\mathfrak{se}(3)$ that projects the spatial accelerations onto the admissible subspace of joint accelerations.

The acceleration of a robotic system is defined as

$$\mathbf{A} = \dot{\mathbf{V}} = \mathbf{J}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}}\dot{\boldsymbol{\theta}}.$$

The previous expression is summarized by equation (6.33), which depends on (6.32). From these expressions, the joint accelerations can be computed in terms of the spatial accelerations of the i -th bodies.

6.3 Control Analysis and Design

The starting point is a PD controller of the form

$$\boldsymbol{\tau} = \mathbf{K}_v(\dot{\boldsymbol{\theta}}_d - \dot{\boldsymbol{\theta}}) + \mathbf{K}_p(\boldsymbol{\theta}_d - \boldsymbol{\theta}).$$

This controller includes proportional and derivative feedback terms based on the tracking error. A PD structure is selected because it introduces both velocity feedback and damping into the reference-tracking process.

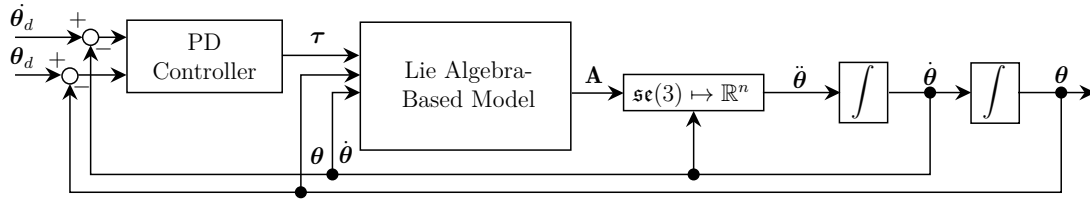


Figure 6.6 / Closed-loop architecture employed for joint trajectory tracking.

The input variables are directly linked to the actuated joints θ_{act} . Therefore, the reference position, velocity, and acceleration assigned to these joints are specified through the following expressions and their corresponding time derivatives:

$$\boldsymbol{\theta}_{d2} = A \sin(2\pi ft + \pi/8), \quad (6.43)$$

$$\boldsymbol{\theta}_{d4} = A \sin(2\pi ft - \pi/8). \quad (6.44)$$

Here, $A = 45^\circ$ represents the amplitude of the signal, $f = 0.5$ Hz denotes the oscillation frequency, and t represents time in seconds. The geometric and physical parameters are taken from [95].

The initial conditions selected for these simulations/scenarios are

$$\boldsymbol{\theta} = \begin{bmatrix} -\frac{\pi}{8} & \frac{\pi}{8} \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = [0 \quad 0].$$

Therefore, by implementing the block diagram presented in Figure 6.6, the closed-loop responses shown in the following figures are obtained.

Simulation No. 1: The results show that the trajectory tracking task is successfully achieved. In addition, the control signal remains within magnitudes that are feasible for implementation using the selected actuators. In this test, only the trajectory tracking performance was evaluated. The obtained results are shown in Fig. 6.7.

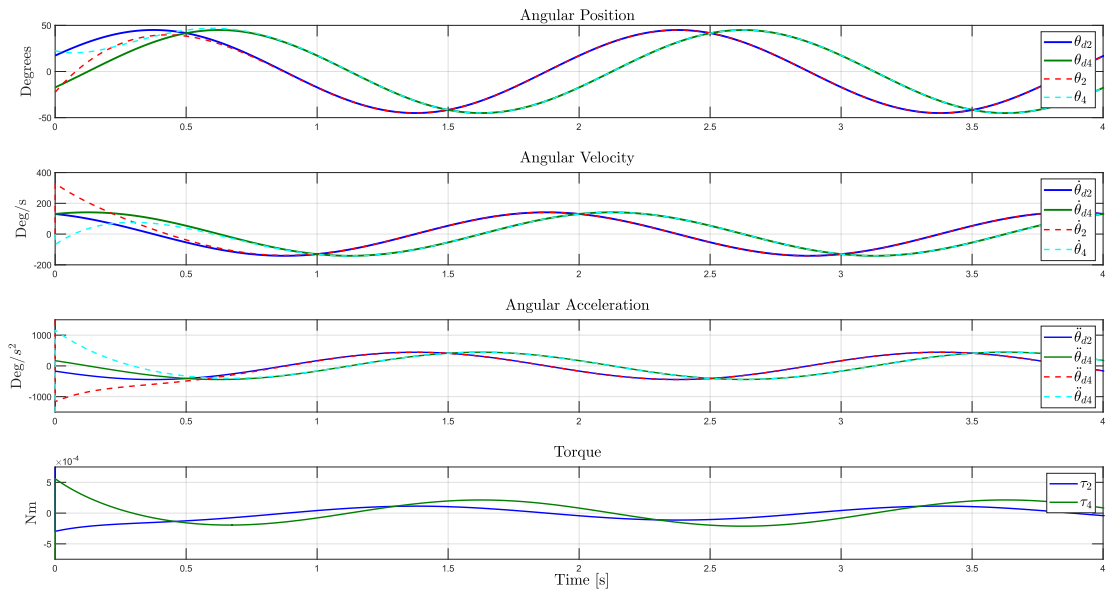


Figure 6.7 / Closed-loop responses with PD control.

Simulation No. 2: In this test, additive disturbances were introduced into the position signals in order to emulate changes in the joint positions caused by external forces. At time $t = 1$ s, a disturbance of magnitude $-\pi/6$ was added and maintained constant. The controller successfully compensates for this perturbation. Subsequently, at $t = 2.5$ s, an additional disturbance with magnitude $\pi/6$ was introduced.

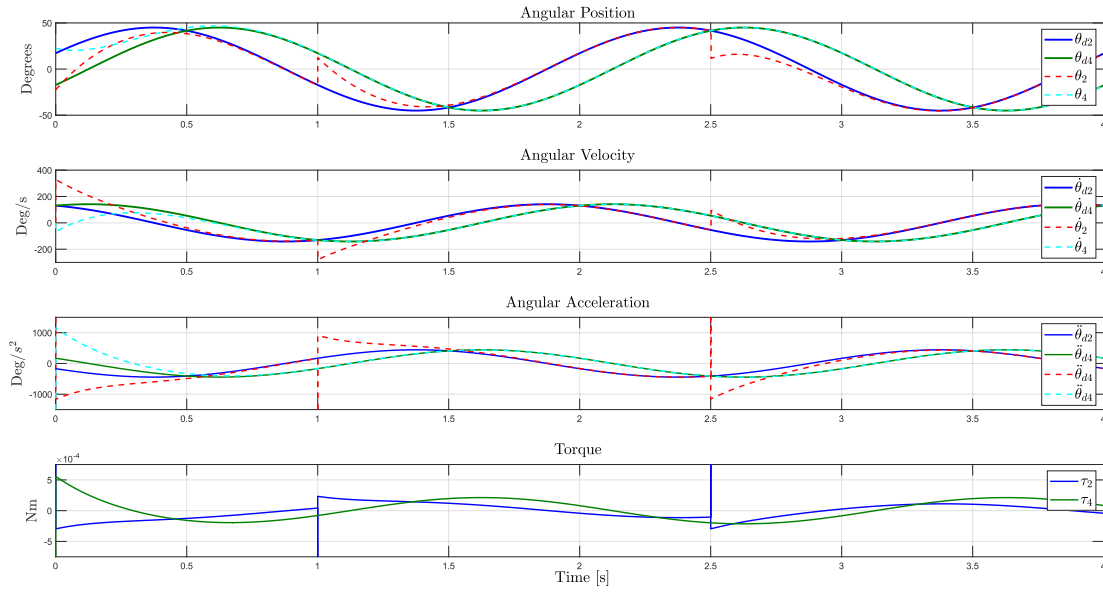


Figure 6.8 / Closed-loop responses with PD control under external disturbances.

Simulation No. 3: In this scenario, a change in the reference frequency is introduced at simulation time $t = 2$ s, modifying the excitation frequency from $f = 0.5$ Hz to $f = 0.75$ Hz. The purpose of this modification is to emulate the desired increase in the joint motion speed. The results demonstrate that the controller maintains accurate trajectory tracking despite the frequency variation.

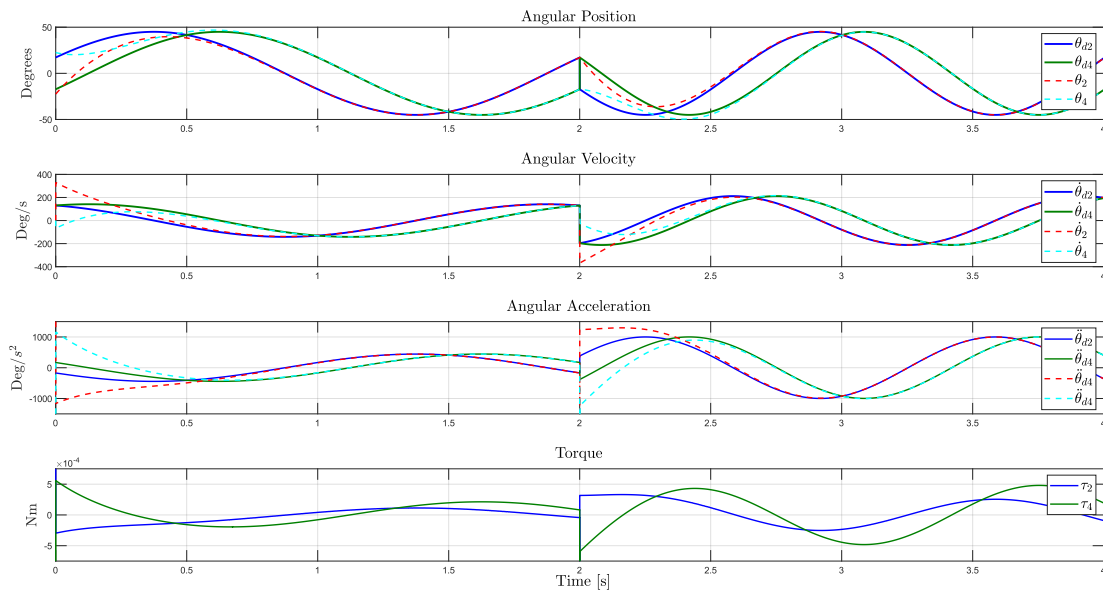


Figure 6.9 / Closed-loop responses under reference frequency variation.

Concluding remarks

Thoroughly conscious ignorance is the prelude to every real advance in science.

James Clerk Maxwell

This thesis focused on the study, synthesis, and optimization of delay-based control strategies for dynamical systems, including both delay-free systems and systems with intrinsic time delays. The proposed methodologies were developed from a frequency-domain perspective using quasi-polynomial analysis, stability crossing boundaries, D-partition techniques, and optimization-based procedures. The obtained results demonstrated that intentional delays can be systematically incorporated into controller structures to improve closed-loop stability, transient performance, robustness, and exponential convergence properties.

First, proportional–integral (PI) controllers for systems with inherent delays were studied. Analytical stability regions were derived from the corresponding characteristic quasi-polynomials through stability crossing techniques. These regions provided a systematic framework for determining admissible controller gains while preserving prescribed stability properties. Furthermore, optimization-based procedures were employed to improve transient performance and exponential decay rates according to criteria associated with spectral abscissa, settling time, and control effort.

Subsequently, the thesis investigated the Proportional Delayed Integral (P δ I) controller for delay-free systems. In this approach, intentional delays were incorporated into the integral action and used as additional tuning parameters. The obtained results showed that suitable delay selections can significantly improve transient response characteristics and achievable exponential decay rates compared with conventional PI structures.

Finally, delay-based controllers for systems with inherent delays were analyzed through the Integral Retarded (IR) and Integral–Proportional Delayed (I–P δ) controllers. Explicit stability conditions and σ -stability regions were derived for general classes of delayed plants. The obtained results demonstrated that delayed control actions can enlarge admissible stability regions and improve robustness properties while preserving closed-loop stability.

Overall, the proposed methodologies provide a systematic framework for the analysis and synthesis of delay-based controllers using analytical stability boundaries and optimization tools. Numerical simulations and experimental validations

confirmed the effectiveness of the proposed approaches and demonstrated the potential advantages of intentionally incorporating delays into controller structures for a wide class of dynamical systems.

Several research directions may be considered as future work. First, the proposed methodologies can be extended to multi-input multi-output systems and higher-order delayed dynamics. Another interesting direction consists in incorporating robustness against parametric uncertainties and external disturbances directly into the optimization framework. Additionally, adaptive or gain-scheduled versions of the proposed controllers could be investigated for nonlinear or time-varying systems. Experimental validation in additional applications, including robotic systems, power electronics, and networked control systems, also constitutes an important continuation of the present work.

Laplace transforms

No.	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \dots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at}, \quad n = 1, 2, 3, \dots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}, \quad n = 1, 2, 3, \dots$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a} (1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a} (e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a} (be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab} \left[1 + \frac{1}{a-b} (be^{-at} - ae^{-bt}) \right]$	$\frac{1}{s(s+a)(s+b)}$

Table A.1 / Basic Laplace transform pairs.

No.	Property
1	$\mathcal{L}\{Af(t)\} = AF(s)$
2	$\mathcal{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$
3	$\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = sF(s) - f(0^+)$
4	$\mathcal{L}\left\{\frac{d^2}{dt^2}f(t)\right\} = s^2F(s) - sf(0^+) - f'(0^+)$
5	$\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} = s^nF(s) - \sum_{k=1}^n s^{k-1}f^{(n-k)}(0^+), \quad f^{(i)}(t) = \frac{d^i}{dt^i}f(t)$
6	$\mathcal{L}_{\pm}\{\int f(t) dt\} = \frac{F(s)}{s} + \frac{1}{s} [\int f(t) dt]_{t=0^{\pm}}$
7	$\mathcal{L}_{\pm}\left\{\underbrace{\int \cdots \int}_n f(t)(dt)^n\right\} = \frac{F(s)}{s^n} + \sum_{k=1}^n \frac{1}{s^{n-k+1}} \left[\underbrace{\int \cdots \int}_k f(t)(dt)^k \right]_{t=0^{\pm}}$
8	$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$
9	$\int_0^{\infty} f(t) dt = \lim_{s \rightarrow 0} F(s), \quad \text{if } \int_0^{\infty} f(t) dt \text{ exists}$
10	$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$
11	$\mathcal{L}\{f(t-a)1(t-a)\} = e^{-as}F(s), \quad a \geq 0$
12	$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$
13	$\mathcal{L}\{t^2f(t)\} = \frac{d^2F(s)}{ds^2}$
14	$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}, \quad n = 1, 2, 3, \dots$
15	$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} F(\sigma) d\sigma, \quad \text{if } \lim_{t \rightarrow 0} \frac{1}{t}f(t) \text{ exists}$
16	$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as)$
17	$\mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau) d\tau\right\} = F_1(s)F_2(s)$
18	$\mathcal{L}\{f(t)g(t)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(p)G(s-p) dp$

Table A.2 / Properties of the Laplace transform.

Initial Value Theorem

$$f(0^+) = \lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

Final Value Theorem

$$f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

Pulse Function

$$f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$$

$$\mathcal{L}\{f(t)\} = \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$$

Impulse Function

$$g(t) = \lim_{t_0 \rightarrow 0} \frac{A}{t_0}, \quad 0 < t < t_0,$$

$$g(t) = 0, \quad t < 0, \quad t_0 < t$$

$$\mathcal{L}\{g(t)\} = \lim_{t_0 \rightarrow 0} \left[\frac{A}{t_0 s} (1 - e^{-st_0}) \right]$$

Applying L'Hôpital's rule,

$$= \lim_{t_0 \rightarrow 0} \frac{\frac{d}{dt_0} [A(1 - e^{-st_0})]}{\frac{d}{dt_0} (t_0 s)}$$

$$= \lim_{t_0 \rightarrow 0} \frac{Ase^{-st_0}}{s} = A$$

Table A.3 / Additional Laplace transform theorems and special functions.

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